Study of cutting the Möbius

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ABSTRACT: The paradox of cutting the Möbius strip is that longitudinal cutting is ambiguous, unlike a cylindrical surface. The configurations and geometrical dimensions of the obtained strips depend on the distance from the cut to the edge of the surface. For example, if there was cutting into one-sixth of the width of a cylindrical surface, it will result in two separate cylindrical surfaces of different widths. This is different with a Möbius strip: if there was cutting into one-sixth of the width of a Möbius surface, it produces two connected strips. One of the strips is twice as large as the original Möbius strip and has 3 twists, and the second one is the same as initial Möbius strip. To date, the parameters and characteristics of the Möbius strip have been reliably identified, which determine the ambiguity of the cutting result, unlike a cylindrical surface.

The interest in the cutting paradox is of both scientific and of practical interest for cardiac surgery. If we manage to resolve this paradox, that is, to find out what parameters and characteristics lead to the ambiguity of the cutting, this will solve one of the problems in cardiac surgery.

Thus, the problem (of which the solution is described in this article) is the identification, experimentally and theoretically, of factors that determine the uniqueness of the configurations obtained by cutting a Möbius strip and studying the resulting configurations

KEYWORDS: Topology; Cylindrical surface; Möbius Strip; One-sided Surface; Surface cutting

Introduction. The study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures is called Topology. In this article we will discuss the study of geometric properties of a Möbius strip. The Möbius strip, also called the Möbius loop is a model that can be obtained by turning a long strip of paper a half turn and then connecting its ends together.[1]

The father of the Möbius strip is August Ferdinand Möbius, a student of Gauss. He wrote numerous works on geometry but became famous for the discovery of a one-sided surface in 1858.

Cylindrical Surface and Möbius Strip. For a better understanding of the differences in the Möbius strip from the usual cylindrical surface, it is necessary to directly compare them.

If one puts a finger on the side of the cylindrical surface and moves it endlessly without lifting one's finger from the surface, the movement will take place only on one side. To get to the other side of a cylindrical surface, there is need to remove one's finger and put it on the second side. This is different with a Möbius strip; if one puts a finger on the side of the cylindrical surface and moves it endlessly without lifting one's finger from the surface, the movement will take place on both sides of surface. It means that the cylindrical surface is double-sided and has a discontinuous surface while the Möbius strip is one-sided and with a continuous surface.

Also, the surface is orientable if it has two sides. As already known, the cylindrical surface has two sides, so it is orientable and the Möbius surface is not orientable.

From the comparison it can be seen that the cylindrical surface and the Möbius surface are completely different geometric objects, although they are made of a similar long strip.

Cutting a Cylindrical Surface and Möbius Strip. As is already known, any longitudinal cutting of a cylindrical surface gives two cylinders of different heights. Subsequent to the longitudinal cutting of a Möbius strip, the resulting shapes have completely different configurations depending on the distance from the scissors to the edge of the strip.[2] This paradox is especially interesting for many topologists.[3]

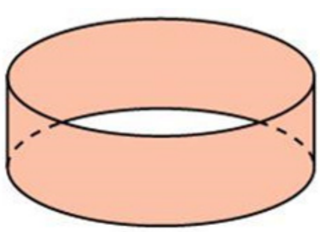

Figure 1. The image of a cylindrical surface.

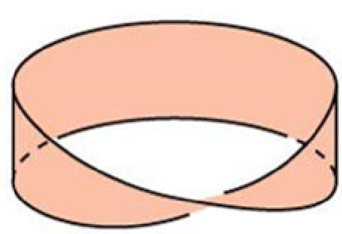

Figure 2. The image of the Möbius strip.

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Characteristics of the cylindrical surface	Characteristics of the Möbius strip
Double-sided surface	One-sided surface
Noncontinuity	Continuity
Orientable	Not orientable

Table 1. Geometrical and topologial characteristics of the cylindrical surface and Möbius strip.

This experiment was repeated to compare the obtained shapes in the longitudinal cutting of a cylindrical surface and a Möbius strip. The researcher hypothesizes that after experiments and studying the obtained geometric figures, the paradox can be explained analytically using the condition of the proportionality between the area (S) of the Möbius strip and width of the Möbius strip (λ), the radius of the circle forming the Möbius strip (R), the condition of the proportionality between the length of the edge (L) and the width of the Möbius strip (λ), as well as radius of the circle forming the Möbius strip (R).

Methods. To begin with, initial samples of a cylindrical surface and a Möbius strip were made from strips of paper of equal length and width (Figure 3).[4] For ease of cutting and making models, striped paper was used.

Figure 3. Original surfaces: Cylindrical surface and Möbius strip.

Experiment #1. After preparing the models, the longitudinal cutting of the two surfaces begins by cutting into one-sixth of the initial width. As can be seen from the experiment, two cylindrical surfaces of different widths were obtained by longitudinal cutting of the cylindrical surface. By splitting the Möbius surface longitudinally, two connected strips were obtained: one twice as large as the original Möbius strip and with 3 twists, and one initial Möbius strip (Figure 4).

Figure 4. Experimental results in longitudinal cutting of the cylindrical surface and Möbius surface by one-sixth of the initial strip width.

Experiment #2. Next, we increase the cut strip to one-fifth of the initial width of the strip. From the experiment, two cylindrical surfaces of different widths were obtained by the longitudinal cutting of the cylindrical surface. At longitudinal cutting of the Möbius surface, two connected tapes were obtained: one twice as large as the original Möbius strip and with 2 twists, and one Möbius strip of the original size (Figure 5).

Figure 5. Experimental results in longitudinal cutting of the cylindrical surface and Möbius surface by one-fifth of the initial width of the strip.

Experiment #3. Next, we cut the cylindrical surface and the Möbius surface longitudinally, retreating from the edge of the strip by one-fourth of the original strip width. From the experimental results it can be seen that cutting a cylindrical surface, two new cylindrical surfaces of different heights were obtained. When cutting the Möbius surface, two connected strips were obtained again: one twice as large as the original Möbius strip and with 2 twists, and one Möbius strip of the original size.

Figure 6. Experimental results in longitudinal cutting of the cylindrical surface and Möbius surface by one-fourth of the initial strip width.

Experiment #4. Next, we cut the cylindrical surface and the Möbius surface longitudinally, retreating from the edge of the strip by one-third of the initial strip width. From the experimental results it can be seen that cutting the cylindrical surface, two new cylindrical surfaces of different heights were obtained. When cutting the Möbius surface, two connected strips were obtained: one twice as large as the original Möbius strip and with 3 twists, and one Möbius strip of the original size.

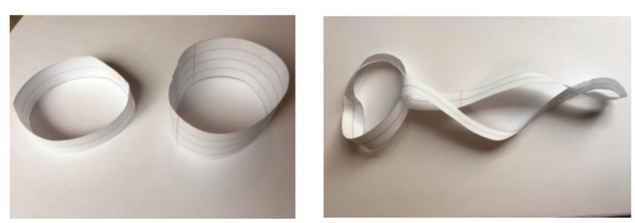

Figure 7. Experimental results in longitudinal cutting of the cylindrical surface and Möbius surface by one-third of the original strip width.

Experiment #5. In the last experiment, we cut the cylindrical surface and the Möbius surface in half, lengthwise. From

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the results of the experiment it is seen that by cutting a cylindrical surface in half, two equal cylinders were obtained. When cutting the Möbius surface in half, one surface was obtained. This surface is twice in length of the original Möbius surface and has 4 twists.

Figure 8. Experimental results in longitudinal cutting of the cylindrical surface and Möbius surface in half.

Results and Discussion. Following the results of five experiments with cutting a cylindrical surface and Möbius surface, the summarizing result is shown in Table 2.

Experiment Results # 1-5

№	Experiment Description	The result of cutting the cylindrical surface	The result of cutting the Möbius surface
1	Cut into one-sixth	Two cylindrical surfaces of different heights	Two connected strips: one twice as large as the Möbius strip and with 3 twists, and one Möbius strip of the original size
2	Cut into one-fifth	Two cylindrical surfaces of different heights	Two connected strips: one twice as large as the Möbius strip and with 2 twists, and one Möbius strip of the original size
3	Cut into one-fourth	Two cylindrical surfaces of different heights	Two connected strips: one twice as large as the Möbius strip and with 2 twists, and one Möbius strip of the original size
4	Cut into one-third	Two cylindrical surfaces of different heights	Two connected strips: one twice as large as the Möbius strip and with 3 twists, and one Möbius strip of the original size
5	Cut in half	Two cylindrical surfaces of equal height	One strip, twice the original with 4 twists

Table 2. Table of the experimental results of cutting the cylindrical surface and the Möbius surface in different ways.

From the table, it is possible to predict what will happen with the longitudinal cutting of a cylindrical surface. It is impossible to guess what happens with different longitudinal cutting of the Möbius strip.

Explanation of the Paradox of Cutting a Möbius Strip. Analyzing the obtained experimental results, there is the question of the nature of this geometric topology paradox in experiments with the longitudinal cutting of the Möbius strip. One of the explanations for this paradox is to apply a condition of the proportionality between the area (S) of the Möbius strip and the width of the Möbius strip (λ), the radius of the circle forming the Möbius strip (R), and the condition of the proportionality between the length of the edge of the Möbius strip (L) and width of the Möbius strip (λ), and the radius of the circle forming the Möbius strip (R). These dependencies are obtained analytically (for the case λ << R).

Calculating the integral of *d*L, find the length of the edge:

$$L = L(\lambda) = 2R \int_{-1}^{1} \sqrt{\mu^2 + 4(1+\mu x)^2}\,\frac{dx}{\sqrt{1-x^2}} \qquad (1)$$

where $\mu = \frac{\lambda}{R}$. In particular, consider $\mu \ll 1$. Therefore,

$$L \approx 4\pi R\left(1 + \frac{\lambda^2}{8R^2}\right) \qquad (2)$$

The area of the Möbius strip is determined by the formula:

$$S = \int_{-\lambda}^{\lambda} d\lambda \int_0^{2\pi} \sqrt{\frac{\lambda^2}{4} + \left(R + \lambda \sin\frac{\varphi}{2}\right)^2}\, d\varphi \qquad (3)$$

where (λ, φ) - orthogonal coordinates.

Comparing (1) and (3), can see that $S = \int_0^{\lambda} L(\lambda)d\lambda$, where L(λ) is determined in equation (1). In particular, in the limit of $\lambda \ll R$ (based on equation (2)) find:

$$S \approx 4\pi R\lambda\left(1 + \frac{\lambda^2}{24R^2}\right) \qquad (4)$$

If expression for area of Möbius strip presented as a sum of two terms: $S \approx 4\pi R\lambda + \frac{\pi\lambda^3}{6R}$, first linear term means area of original tape, which length is 2πR and width is 2λ and cubic term means the curvature of the surface.

Expressions (2) and (4) allow for the comparison of the geometric characteristics of the cylindrical surface and the Möbius surface having same cross section – circle of radius R.

Writing the conditions for preserving the length of the edge of the strip and the area of the strip for cylindrical surface and Möbius surface, one obtains a system of equations of the third degree with the number of unknowns equal to the number of bands which are obtained after cutting.

Figure 9. A cardiac muscle patch is used to treat myocardial infarction. It helps to restore damaged heart tissue.

The Impact of Cardiac Surgery. In cardiac surgery, there is a problem of determining the size, shape, and thickness of the material for the cardiac muscle patch, which is attached to the myocardium in the place of a scar from a myocardial infarction (Figure 9).

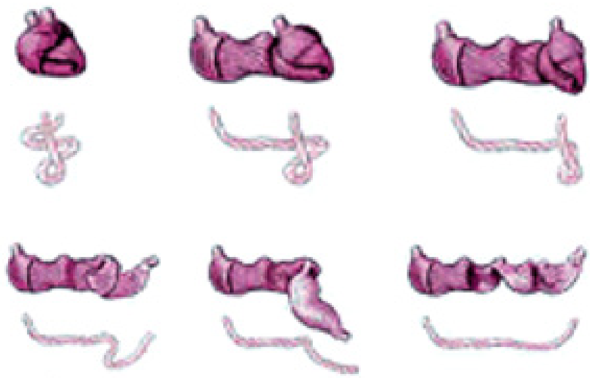

Figure 10. Myocardium is a curved-twisted strip with the topology of a Möbius Strip.

In accordance with the discovery of Dr. Francisco Torrent-Guasp and Dr. Gerald Buckberg myocardium is a thick curved-twisted strip with a Möbius strip topology (Figure 10).[5-6]

This cardiological discovery complicates the issue of determining the configuration of the patch (on the curved-twisted surface with the topology of a Möbius strip). During design and creating the cardiac muscle patch of a patient, the surgeon should consider ambiguity of the incision on the heart muscle. Ignoring this increases the risk of a patient's postoperative pathologies like aneurysm. The factor influencing the solution of this question is the paradox of cutting a Möbius strip.

Therefore, the results of the research described in this article can be applied in cardiac surgery to determine the configuration and size of the cardiac muscle patch.

Conclusion. This article describes the experimental and theoretical studies of the paradox of cutting a Möbius strip; comparison with the cutting of the cylindrical surface, and the generalization of the results. The consequence of this is the identified characteristics and parameters of the Möbius strip, the values of which determine the configuration and the geometric dimensions of the resulting figures after longitudinal cutting. Significant findings from experiments and analysis include the width of the cut strip during the longitudinal cutting of the Möbius strip.

The results of these studies have practical cardiac surgery significance.

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