RESEARCH ARTICLE

A Hypercube unfolding that Tiles R^2 (and R^3)

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ABSTRACT: In their pioneering work, G. Diaz and J. O'Rourke[1] show that the four-dimensional hypercube has an unfolding that tiles three-dimensional space, and can be further unfolded to tile two-dimensional space. This paper presents an alternative path to unfolding the hypercube to tile lower-dimensional space.

KEYWORDS: Mathematics; Geometry and Topology; Computational Geometry; Dimension Descending Tilers; Hypercube Unfoldings; Octocube Unfoldings.

Introduction

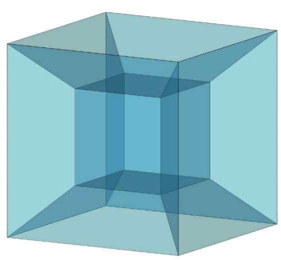

Figure 1: The 4-D hypercube's shadow in 3-D.

The hypercube is the 4-dimensional analog of the cube (Figure 1). Unfolding the hypercube will give its 3-D net like in Figure 2(a); analogous to how unfolding a cube will give its 2-D net (Figure 2(b)).

A 3-D cube has 11 distinct nets and each net is a monohedral tile (monohedral, a tessellation where each tile is congruent). A beautiful pattern emerges, the cube (that obviously tiles $\mathbb{R}^3$) has unfoldings that tile $\mathbb{R}^2$. This entitles the cube to be a dimension descending tiler (DDT).

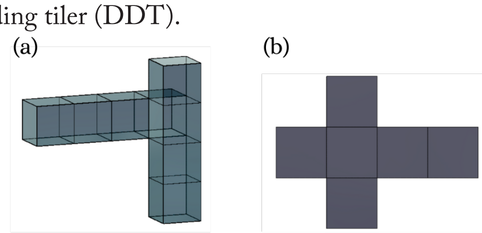

Figure 2: (a) A net of the Hypercube· (b) A net of the cube.

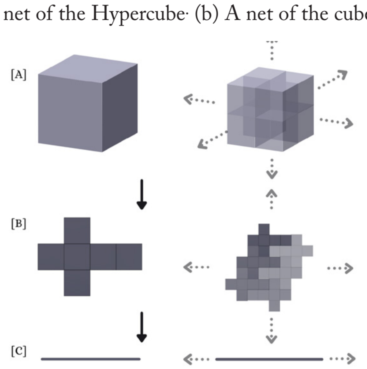

Figure 3: A DDT descending dimensions. (Let an alphabet mark the rung on the ladder) The DDT at [A], its net down a dimension at [B] and so on.

The hypercube – which itself tiles 4-D space – has 261 distinct 3-D nets that tile $\mathbb{R}^3$,[2] analogous to the previous example of the 3-D cube. *Following this pattern, will these 3D nets, in turn, have edge-unfoldings that will tile* $\mathbb{R}^2$?

Diaz and O'Rourke showed the hypercube indeed has a net (that tiles $\mathbb{R}^3$) whose edge-unfolding (an unfolding that cuts along the edges of the polycube) perfectly tiles $\mathbb{R}^2$. Stefan Langerman and Andrew Winslow also showed that the 'Dali Cross' – one of the 261 unfoldings of the hypercube – has an edge-unfolding that tiles the plane.[3] Thus, showing the tesseract to be a dimension descending tiler, from $\mathbb{R}^4$ to $\mathbb{R}^1$. {Note that the unfolding of the 2-D net is a line that obviously tiles $\mathbb{R}^1$} This paper explores another path to unfold the 4-D hypercube to tile lower-dimensional spaces.

Results and Discussion

A 3-D cube has 6 faces and its net is a 6-square polysquare (refer to Figure 2); similarly, the tesseract has 8 cubes, and its net is an 8-cube polycube, termed an octocube (In this paper, an octocube specifically refers to the 8-cube polycube face-unfolding of the hypercube). All 261 octocubes have been shown to tile $\mathbb{R}^3$. Of which, this paper will focus on the edge-unfolding of octocube #72. (The octocubes are numbered according to Moritz Firsching's list of octocubes).[4] Do note that octocube #72 is quite similar to octocube #120 in Figure 2 of Diaz and O'Rourke's paper.

Net of the hypercube tiling R^3:

First, it can be observed that octocube #72 tiles $\mathbb{R}^3$. The proof for this tiling was first shown by Jose A. Gonzalez.[5] This paper will briefly describe how octocube #72 tiles $\mathbb{R}^3$, in a simpler but tweaked way. The following stacking procedure is very similar to that in Figures 3 and 4 of Diaz and O'Rourke's paper.

1. First it is observed that the octocubes can be 'inserted' into each other to form a stack of 3 octocubes (Figure 4). This forms a uniform 'staircase' on the back left and back right side (back right side not visible in Figure 4), and the front bottom and front top sides. {Of course, choosing three octocubes is arbitrary. Three were chosen for simplicity; it can be imagined as an infinite number of octocubes to be stacked in such a way to form an infinite diagonal from south-west to north-east}.

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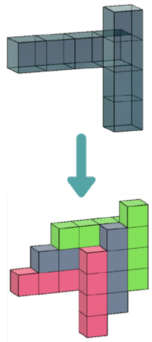

Figure 4: 3 stacked octocube #72s.

2. Stack two of these structures to get the structure in Figure 5. Here, the red block's back left staircase is nestled into its 'negative' in the blue stack (present behind and below, not visible in Figure 5). Again, an infinite diagonal of this structure can be imagined.

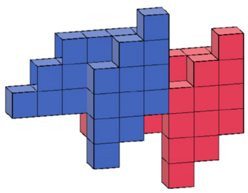

Figure 5: Stacked triplets of octocube #72s.

3. A larger portion of the infinite diagonal stack is shown in Figure 6. Stacking this infinite diagonal on the top and the bottom, infinite times, will result in a 2-cube thick slab that runs forever in the X-Z plane. Just as in Figure 4 of Diaz and O'Rourke's paper.

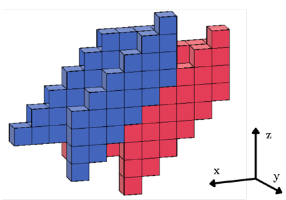

Figure 6: A portion of the infinite diagonal stack.

4. It is obvious that layering these 2-cube thick slabs (in the y-direction) will tile $\mathbb{R}^3$ perfectly. Hence proved the octocube tiles 3-dimensional space (proved first by Jose A. Gonzalez).

This is the first descent down a dimension.

The net of Octocube #72:

Now it is necessary to introduce the net of this Octocube that will tile 2D space. This is the second descent down a dimension.

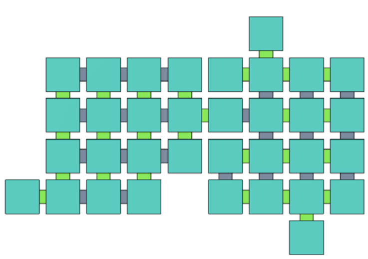

Figure 5: Net of octocube #72. The green joints between squares are all valley-folds, and the blue joints do not hinge.

A step-by-step transformation of the net of octocube #72 is shown in Figure 8.

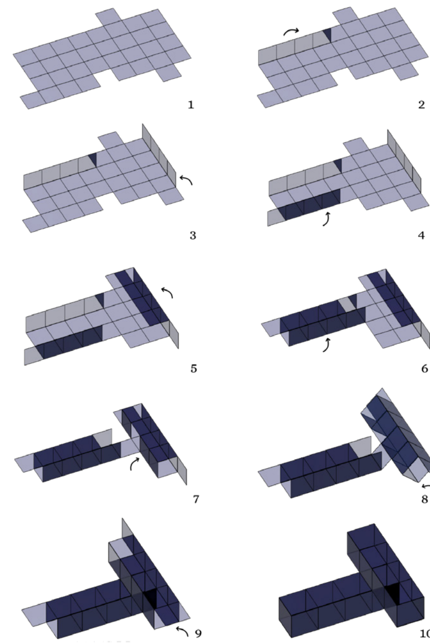

Figure 8: A step-by-step transformation of the net to octocube #72.

Not all nets are Valid:

All octocubes tile $\mathbb{R}^3$, so do all the unfoldings of an octocube tile $\mathbb{R}^2$? The answer is no. Not all nets of octocube #72 tile $\mathbb{R}^2$. The following are (two of many) valid nets, but do not tile $\mathbb{R}^2$ (Figures 9 and 10).

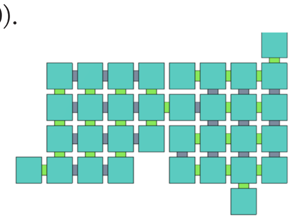

Figure 9: A slight modification of the net in Figure 7, that makes it not tile $\mathbb{R}^2$.

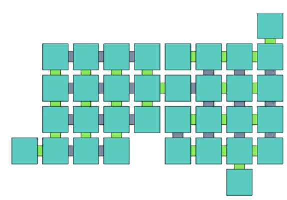

Figure 10: Another net of octocube #72 that does not tile $\mathbb{R}^2$.

Net of the Octocube #72 tiling R^2:

The net of octocube #72 tiling $\mathbb{R}^2$ is shown in Figure 11.

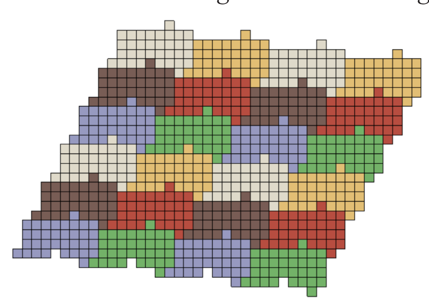

Figure 11: A net of the octocube shown tiling $\mathbb{R}^2$.

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Conclusions

The paper has presented an alternative path than what has been by G. Diaz, J. O'Rourke, and S. Langerman, and A. Winslow to unfolding the hypercube to tile lower-dimensional space. However, it is still left to answer (besides what has not been answered in Diaz and O'Rourke's paper):

• How many nets of an octocube are there that do tile $\mathbb{R}^2$; or does only one net tile $\mathbb{R}^2$?

• What is the ratio between the nets that do and those that do not tile $\mathbb{R}^2$?

• What is this ratio when a higher dimensional DDT (if they exist) descends the ladder (the ladder as was in Figure 3)?

Acknowledgements

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Author

Trun loves physics and recreational math. He is passionate about exploring new ideas and breaking barriers. Being an avid inventor and entrepreneur, he has ventured into advancing miniature satellite technology. Trun aspires to advance Quantum Computing to accelerate technological development and solve the world's biggest challenges.