RESEARCH ARTICLE

Simulation of Heat Diffusion in a Tidally-Locked Exoplanet

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ABSTRACT: I perform a numerical calculation of the heat diffusion in a simulated tidally-locked exoplanet. I solve the heat diffusion equation for the surface and the interior of the planet using a finite element grid. I develop a C++ code specifically for this purpose. I take into account both the energy received from the star, the energy radiated into space, the heat source provided by the radioactive decay inside the planet, and the effects of a single-layer atmosphere model. I apply the simulation to the case of Proxima Centauri b. The evolution of the temperature distribution is obtained in the time range from 0 to 15 Gyr. The most important results are the prediction of the possible formation of underground oceans in the dark side and the identification of the habitable zone on the surface of the exoplanet.

KEYWORDS: Physics and Astronomy; Computational Physics; Exoplanets; C++ Simulation; Heat Diffusion.

Introduction

There is a growing number of exoplanets that are being discovered. Many of them are rocky planets, similar to Earth, that orbit in the habitable zone. This is a region around a star in which an exoplanet can host liquid water on its surface and possibly support life.[1,2] The presence of liquid water is possible only at a narrow range of temperatures. On the Earth's surface, at atmospheric pressure of 10^5 Pa, this range is from about 270 to 370 K, but it may vary if the pressure of the explanet's atmopshere is significantly different.[3] Understanding the thermal evolution of an exoplanet is fundamental to determining where and when the conditions for habitable zones are satisfied.

The temperature reached by a planet during its evolution depends on several factors: it absorbs a part of the electromagnetic radiation emitted by its star; it produces internal heat with radioactive decay in the planet's core; it diffuses a part of its heat into space, according to the Stefan-Boltzmann law.

A critical parameter is the planetary equilibrium temperature, which is the theoretical temperature that a planet would reach if it were a black body. Thermal equilibrium induced by radiation is reached when the power transmitted by the star to the planet will be equal to that lost by the planet in space. The temperature, T_{eq}, at which this equilibrium is reached, averaged over the planet's surface, is equal to:

$$T_{eq}=\left[\frac{L_{\star}}{4\pi D^2}\cdot\frac{(1-a)}{4\sigma}\right]^{1/4} \tag{1}$$

where a is the albedo, L_* is the star luminosity, σ is the Stefan-Boltzmann constant, and D is the orbit semi-axis.[4] The albedo is the fraction of the star radiation reflected into space by clouds, sea-ice, and other bodies on the planet's surface.

The planet's atmosphere plays an essential role for two reasons. First, it contributes to raising the average temperature due to the greenhouse effect. Second, it contributes with the winds to transport heat between the illuminated face and the dark one, partly rebalancing the distribution of surface temperature.[5]

I considered a simplified scenario of a single-layer atmosphere model[6] that I slightly altered to calculate the total energy budget as shown in Figure 1. At the equilibrium.

$$\begin{aligned}\sigma T_{eq}^4+\epsilon_{atm}\sigma T_{atm}^4&=\epsilon_p\sigma T_p^4\\ \epsilon_p\sigma T_p^4&=(1-\epsilon_{atm})\epsilon_p\sigma T_p^4+2\epsilon_{atm}\sigma T_{atm}^4,\end{aligned} \tag{2}$$

where the planet and the atmosphere are supposed to be gray bodies with emissivities of ϵ_P and ϵ_{atm} , respectively.

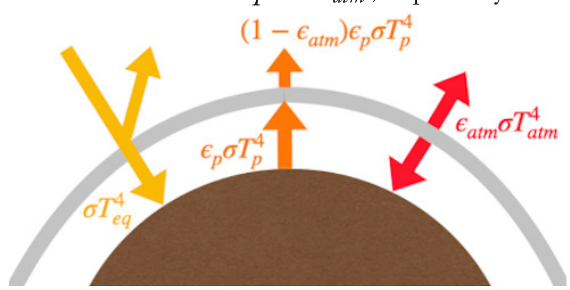

Figure 1:

Thus, the temperatures of the atmosphere and the planet are expected to be

$$\begin{aligned}T_{atm}&=(2-\epsilon_{atm})^{-1/4}T_{eq}\\ T_p&=\left[\epsilon_p\left(1-\frac{\epsilon_{atm}}{2}\right)\right]^{-1/4}T_{eq}.\end{aligned} \tag{3}$$

In fact, from the second equation in Eq. 2, it follows that $\epsilon_P T_P^4=2T_{atm}^4$. Substituting this relation in the first equation, it results in $T_{atm}^4=(2-\epsilon_{atm})\ T_{eq}^4$, from which the first of Eq. 3 is derived. By eliminating T_{atm}^4 from the first equation in Eq. 2 and rearranging the terms, it follows T_eq^4(1+ε_atm/(2-ε_atm)) =ε_pT_p^4 $/(\epsilon_P\ (2\epsilon_{atm}))\ T_{\mathrm{eq}}^4$, from which the second equation in Eq. 3 is obtained.

Note that T_{eq}, as defined in Eq. 1, appears in Eq. 3 only as a reference for the average incoming stellar radiation flux, but in general neither the atmosphere nor the planet's surface is

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at a temperature T_{eq} at the equilibrium. In the limit $\epsilon_{atm}\rightarrow 1$, the atmosphere temperature could approach the equilibrium temperature T_{eq}. However, because of the greenhouse effect, the temperature of the planet's surface T_P remains always greater than T_{eq} if $\epsilon_{atm}>0$, even if $\epsilon_{atm}\rightarrow 1$.

In this paper, I present a simulation of the heat diffusion for the exoplanet Proxima Centauri b. It orbits around Proxima Centauri, a red dwarf star with a mass of about 12.5% of the Sun's mass and luminosity of about only 0.1% that of the Sun. Proxima Centauri b is an exoplanet very similar to Earth located at only 0.0485 UA. Proxima Centauri b and its star form an isolated star-planet system subjected only to gravitational tides. Therefore, there is an high possibility that Proxima Centauri b is tidally locked in a nearly circular orbit.[7]

Proxima Centauri b is located in the habitable zone[8] and, considering an albedo of 0.3, it has an equilibrium temperature of 229 K. Assuming a planet emissivity of 0.9, and an atmosphere emissivity of 0.9, the temperature of the planet is 273 K and the temperature of the atmosphere is 223 K.

Methods

Spherical coordinates:

Starting from Fourier's law of heat conduction, I obtained the heat diffusion equation in spherical coordinates[9]:

$$q=-kA\frac{dT}{dL}, \tag{4}$$

where q is the heat flux in $[W]$ through the surface A in $[m^2]$, k is the material conductivity in$[W/(m\cdot K)]$, and dT/dL is the temperature gradient perpendicular to the surface.

I chose to represent the temperature on a grid of nodes in spherical coordinates. Using Eq. 4, I calculated the heat flux through the faces of an infinitesimal cell centered in r,ϕ,θ, (see Figure 2) obtaining the following

$$\begin{aligned} q_r &= -k r^2 d\phi \sin\theta d\theta \frac{\partial T}{\partial r} \\ q_\phi &= -k\frac{dr\, d\theta}{\sin\theta}\frac{\partial T}{\partial \phi} \\ q_\theta &= -k\, dr \sin\theta\, d\phi \frac{\partial T}{\partial \theta} \end{aligned} \tag{5}$$

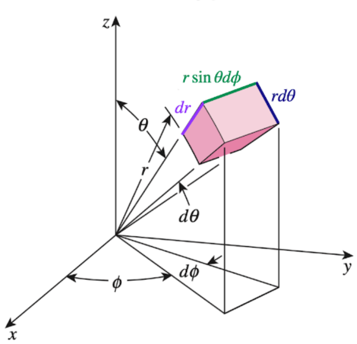

Figure 2:

Considering a heat capacity c_P and a density ρ, from

$$dQ=c_p m\, dT \tag{6}$$

Where dQ is the amount of heat that must be added (at constant pressure) to an object of mass m to raise its temperature by dT, I obtained

$$q=c_p\rho V\frac{\partial T}{\partial t}, \tag{7}$$

where $V=r^2 \sin\theta dr d\phi d\theta$.

Developing in the Taylor series q_{r+dr} up to the first order

$$q_{r+dr}=q_r+\frac{\partial q_r}{\partial r}dr \tag{8}$$

The net heat flux in the cell is:

$$q_{cell}=q_{r+dr}-q_r=\frac{\partial q_r}{\partial r}dr. \tag{9}$$

By repeating for all directions and by adding the three contributions, using Eq. 5 and Eq. 7, I solved for the heat diffusion equation in spherical coordinates

$$\begin{aligned} \frac{\partial T}{\partial t} &= \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 T}{\partial\phi^2}+\right. \\ &\left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)\right]+\frac{Q_{decay}}{c_p\rho}, \end{aligned} \tag{10}$$

where $\alpha=k/c_p\rho$ is the diffusivity constant in $[m^2/s]$ and Q_{decay} is the power generated per unit volume by the internal nuclear decays in $[W/m^3]$.[10]

I further developed the derivatives of the product in r and θ re-writing:

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) &= \frac{\partial r^2}{\partial r}\frac{\partial T}{\partial r}+r^2\frac{\partial^2 T}{\partial r^2}= \\ &=2r\frac{\partial T}{\partial r}+r^2\frac{\partial^2 T}{\partial r^2}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) &= \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial T}{\partial\theta}+\sin\theta\frac{\partial^2 T}{\partial\theta^2}= \\ &=\cos\theta\frac{\partial T}{\partial\theta}+\sin\theta\frac{\partial^2 T}{\partial\theta^2}. \end{aligned} \tag{12}$$

I solved Eq. 10 using a finite element grid. I developed a dedicated C++ code specifically for this purpose.

Each cell of the grid is centered in a node at position (r,ϕ,θ) and has a volume of $\Delta V=\Delta r\cdot r\sin\theta\Delta\phi\cdot r\Delta\theta$.

Indeed, to solve Eq. 10 it is necessary to have both first and second derivatives. Therefore, I developed the temperature in the Taylor series up to the second order. For example, let's consider the radial direction

$$\begin{aligned} T(r+\Delta r) &= T(r)+\frac{\partial T}{\partial r}\Delta r+\frac{\partial^2 T}{2\partial r^2}\Delta r^2 \\ T(r-\Delta r) &= T(r)-\frac{\partial T}{\partial r}\Delta r+\frac{\partial^2 T}{2\partial r^2}\Delta r^2 \end{aligned} \tag{13}$$

Adding and subtracting, I found the first and second derivatives

$$\begin{aligned} \frac{\partial T}{\partial r} &= \frac{T(r+\Delta r)-T(r-\Delta r)}{2\Delta r} \\ \frac{\partial^2 T}{\partial r^2} &= \frac{T(r+\Delta r)+T(r-\Delta r)-2T(r)}{\Delta r^2}. \end{aligned} \tag{14}$$

These formulas allow me to compute the derivatives of the temperature for each node of the cell considering the temperatures of the node before and of the node after, along each direction.

Time Evolution :

The evolution of temperature is obtained by discretizing the time in steps of finite duration Δt. I first calculated the spatial derivatives at a given time and then I use Euler's formula to extrapolate the temperature at the next time step

$$T(t+\Delta t)=T(t)+\frac{\partial T}{\partial t}\Delta t+\ldots, \qquad (15)$$

where the time derivative at a time t is provided by Eq. 10.

Euler's method is known to be unstable, so the time step Δt is chosen adaptively such that at each time the maximum variation of temperature is 10%.

One-Dimension test :

I tested the code in the one-dimensional case along the planet's equator. Only heat diffusion was taken into account. The initial condition is $T_0(\phi)=(300+20 sin\phi)K$. I used a periodic boundary condition since the equator is a closed ring of radius r. In this simple situation, it is possible to derive an analytical solution to Eq. 10: $T(\phi,t)=(300+20 sin\phi e^{-t/t_c})K$, where the characteristic time is given by $t_c=r^2/\alpha$. By assuming $\alpha=2$ m^2/s, $d\phi=0.36°$, and $r=1000 Km$, $t_c=15 Gyr$.

In Figure 3 I show the comparison of the numerical results (red dots) with the analytical solution (blue lines).

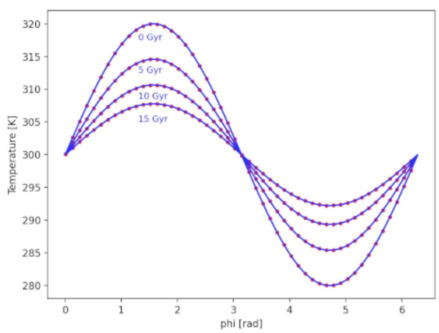

Figure 3: 1D simulation VS analytical solution.

Three-Dimension Simulation:

I applied the numerical method described in the section above to the case of the exoplanet Proxima Centauri b. I performed a three-dimension simulation adopting the physical parameters listed in Table 1.

For decay power, density, thermal conductivity, and specific heat capacity I took Earth's average values.[11] For simplicity, I assumed that all physical parameters are uniform in space and constant in time, except for the radioactive decay that decreases with time according to

$$Q_{decay}=0.01 e^{(-t/t_{decay})}[\mu W/m^3]. \qquad (16)$$

The decay of radioactive isotopes like urainum-235 (235U) contributed a large fraction of radiogenic heat to the early Earth due to its relatively short half-live,$t_{1/2}$. Indeed, for this simulation, I assumed $t_{decay}=1 Gyr$ by considering that for 235U $t_{1/2}=0.7 Gyr$ and that $t_{decay}=t_{1/2}/ln$ (2).

In addition, I considered that the radioactive decay is uniformely distributed in the planet's interior, with no radial dependence. No radioactive decay is considered to be present in the atmosphere.

For the Proxima Centauri planetary system, I considered the parameter values reported in the NASA Exoplanet Archive.[12]

It should be considered that some of the fiducial model parameters used to run the simulation are more uncertain than others. Parameters like, the stellar luminosity, the distance of the planet from the star, and the planet radius are reilabe estimates since they came from astronomical observations of the Proxima Centauri system. Others, like the thermal conductivity, are more uncertain because they depend on the composition of the interior of the exoplanet and are not directly accessible. In these cases, average Earth values are assumed. Furthermore, the assumption of constant composition throught the entire radius of the planet should be considered as a zeroth-order approximation.

The C++ program used for the simulation is named PLANET CODE and is available in GitHub: https://github.com/GiuliaMurgia03/Simulation-of-Heat-Diffusion-in-a-Tidally-Locked-Exoplanet.

Table 1: Simulation set-up.

Property	Value
ρ crust density	5558 Kg/m^3
k thermal Conductivity	5 $W/(Kg \cdot K)$
Initial Q_{decay}	0.01 $\mu W/m^3$
t_{decay}	1 Gyr
c_p specific heat capacity	200 $J/(Kg \cdot K)$
ϵ_p planet emissivity	0.9
ϵ_{atm} atmosphere emissivity	0.9
L_* star luminosity	0.00154 $L_\odot$
r planet radius	6880 Km
D planet Distance	0.0485 UA
a planet Albedo	0.3
Δt time step	<0.01 Myr
Δr	430 Km
$\Delta\phi$	5.6 deg
$\Delta\theta$	5.6 deg
Number of grid nodes	16x64x32
T_{eq} thermal equilibrium temperature	229 K

Initial conditions:

The simulation begins when the planet has already finished the accretion phase by reaching the final radius. I assumed that the planet has been heated by the release of gravitational energy to a uniform temperature of 1000 K.[13]

Boundary conditions:

I used a periodic boundary condition along the longitudinal direction (ϕ). At the north and south poles of the grid, I assumed that there is no heat flux for θ<0 deg and θ>180 deg along the latitude direction. Thus, according to Eq. 4, $\partial T/\partial\theta=0$. Indeed, from Eq. 14 I obtain $T(\theta+\Delta\theta)=T(\theta-\Delta\theta)$. I applied the same condition along the radial direction for r<0 at the center of the planet, obtaining

$T(r-\Delta r)=T(r+\Delta r)$

For the illuminated side, I considered a radiative boundary condition in which

$$-kA\frac{\partial T}{\partial r}=A\sigma\epsilon_p T^4-4A\sigma T_{eq}^4\sin\theta\cos\phi - -\epsilon_{atm}A\sigma T_{atm}^4. \qquad (17)$$

I obtained

$$\frac{\partial T}{\partial r}=\frac{\sigma}{k}\left(4\cos\phi\sin\theta T_{eq}^4+ +\epsilon_{atm}T_{atm}^4-\epsilon_p T^4\right), \qquad (18)$$

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from which I got the radiative boundary condition

$$T(r+\Delta r)=T(r-\Delta r)+ \\ +\frac{2\Delta r\sigma}{k}\left(4\cos\phi\sin\theta T_{eq}^4+\epsilon_{atm}T_{atm}^4-\epsilon_p T^4\right). \quad (19)$$

For the dark side, only the radiation from the atmosphere contributes to the heat flux toward the surface and the second term on the right-hand side of Eq.17 is null.

Results

I presented the evolution of the temperature for Proxima Centauri b in the time range from 0 to 15 Gyr in Figure 4. The simulation run took about a day of computing time on my laptop. The images were obtained with PLANET CODE using a visualization program written in C++ and OpenGL.[14] I show the northern hemisphere of the illuminated side of the planet, with a sector removed to illustrate the interior.

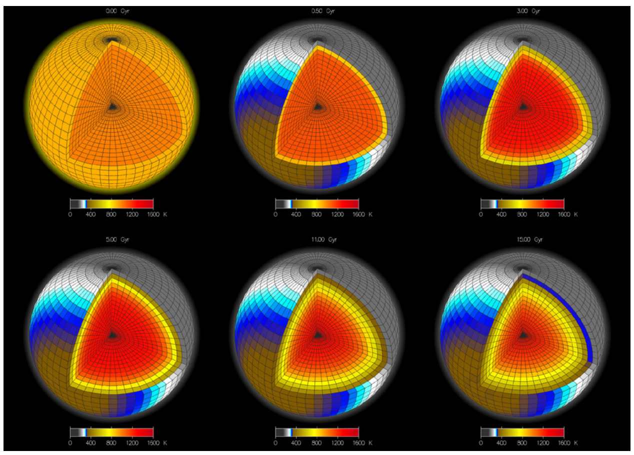

Figure 4: Each cell of the grid is colored based on its temperature as represented in the color bar.

At the beginning (t=0 Gyr), the planet is at an average temperature of 1000 K and the surface is starting to cool down due to the radiative losses. After 0.5 Gyr the surface already reaches its equilibrium temperature, while the interior is warming up due to the heat released by nuclear decay.

Note that at the center of the illuminated side, the temperature is between 313-333 K. Around this central hot area, at mid-latitudes, a ring forms with a temperature between 273 and 313 K. Assuming for the exoplanet an atmospheric pressure of 10^5 Pa, these temperatures permit the existence of liquid water on the surface. Thus, this represents the habitable region of the exoplanet. In the rest of the planet, especially on the dark side, the temperature is far below 273 K and any superficial liquid water would be frozen into ice.

In the period from 0.5 to 3 Gyr, the temperature continues to rise in the core of the planet reaching 1500 K. In the meantime, the internal heat of the planet diffuses by conduction toward the surface, where it is dissipated into space by radiation.

In the period from 5 to 11 Gyr, the hot core gets smaller because the energy source provided by nuclear decay is decreasing exponentially according to Eq. 16. The crust continues to cool deeper and deeper and gets thicker.

The cooling of the crust is more enhanced beneath the dark side and at about 15 Gyr a striking feature appears: just under the dark side surface the temperature would permit the existence of an ocean of warm liquid water that can support lifeforms (at least as we know them here on Earth). Note that, due to the increase of pressure with depth, much hotter liquid water can exist even deeper inside the crust, but this would contrast with the average density of the simulated planet that should be primarly composed of rocks and heavy materials. Furthermore, the hypotetical ocean could be much thinnier than what is represented in the bottom-right image in Figure 4. In fact, the radial resolution of the numerical grid, 430 km, is too coarse to simulate accurately the vertical profile of temperature in these external layers of the planet's crust.

Discussion and Conclusion

It must be considered that this simulation is based on simple assumptions and that some physical mechanisms have not been included. For example, it is assumed that heat transport occurs by conduction within the planet. However, as we know occurs in the Earth's mantle, convection may predominate especially in the presence of moving molten rock. In addition, I used a very simple atmospheric model on the planet's surface with no winds or vertical layers of different temperatures and different chemical compositions.

However, despite its simplicity, the simulation reveals a variety of features. The most important is perhaps the prediction of the presence of underground oceans, like those thought to exist beneath the surface of Jupiter's satellite Europa. The fact that Proxima Centauri b is a tidally-locked planet is essential because this creates an asymmetry in the dissipation of internal heat between the two hemispheres of the planet.

The simulation also makes it clear that for a planet in the habitable zone, such as Proxima Centauri b, it is also very important to consider its orbital parameters. In particular, on a tidally-locked planet, only the mid-latitude regions of the illuminated face can be considered habitable. However, life sustained by liquid water may also be possible on the dark side in conjunction with the presence of underground oceans.

In this simulation, the presence of underground oceans appears at about 15 Gyr. This time is larger than the current age of the Universe. However, this is still compatible with the expected lifetime of low-luminosity red dwarfs stars, like *Proxima Centauri*, which will take trillions of years to burn through their fuel.[15]

It cannot be excluded that an ocean below the surface could form earlier if the cooling is more intense because of a lighter atmosphere or a greater distance between the planet and the star.

In the future, I would like to further develop PLANET CODE including developing in the program a physical simulation of the atmosphere and the water bodies.

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References

1. Kasting J. F., Whitmire D. P., Reynolds R. T. Habitable Zones around Main Sequence Stars. Icarus. 1993, Volume 101, Issue 1, 108-128.
2. Kopparapu R. K., Ramirez R., Kasting J. F. Habitable Zones around Main-sequence Stars: New Estimates. ApJ. 2013, 765, 131.
3. Goldblatt C. Tutorial models of the climate and habitability of Proxima Centauri b: a thin atmosphere is sufficient to distribute heat given low stellar flux. arXiv. 2016, arXiv:1608.07263
4. Michael Strauss. Planetary Thermodynamics. https://www.astro.princeton.edu/~strauss/FRS113/writeup3/ (accessed Dec 10, 2021).
5. Madhusudhan N. Exoplanetary Atmospheres: Key Insights, Challenges, and Prospects. ARA&A. 2019, 57, 617.
6. American Chemical Society, Climate Science Toolkit, Atmospheric Warming, A Single-Layer Atmos-phere Model https://www.acs.org/content/acs/en/climatescience/atmosphericwarming/singlelayermodel.html (ac-cessed Dec 17, 2021)
7. Ribas I., Bolmont E., Selsis F. The habitability of Proxima Centauri b. I. Irradiation, rotation and volatile inventory from formation to the present. A&A. 2016, 596, A111.
8. Barnes, R.; Deitrick, R.; Luger, R. The habitability of Proxima Centauri b I: Evolutionary Scenarios. arXiv:1608.06919, 2018.
9. Wakita, S.; Sekiya, M. Thermal Evolution of Icy Planetesimal in the Solar Nebula. Earth Planet Space. 2011, 63, 1193-1206
10. Wright State CECS. Heat Conduction Equation. https://cecs.wright.edu/~sthomas/htchapter02.pdf (accessed Dec 1, 2021)
11. Whittington, A. G.; Hofmeister, A. M.; Nabelek P. I. Temperature-dependent Thermal Diffusivity of the Earth's Crust and Implications for Magmatism. Nature. 2009, 458, 319-321.
12. NASA EXOPLANET ARCHIVE. Alf Cen Overview. https://exoplanetarchive.ipac.caltech.edu/overview/Proxima%20Cen%20b#planet_Proxima-Cen-b_collapsible (accessed Dec 17, 2021)
13. Elkins-Tanton, L. T. Magma Oceans in the Inner Solar System. Annu. Rev. Earth Planet. Sci. 2012, 40, 113-139
14. Kessenich, J.; Sellers G.; Shreiner D. OpenGL Programming Guide: The Official Guide to Learning OpenGL, 9th ed; version 4.5; Addison-Wesley, 2017.
15. Adams, F.; Laughlin, G.; Graves, G. Red Dwarfs and the End of the Main Sequence. Gravitational Collapse: From Massive Stars to Planets. Revista Mexicana de Astronomía y Astrofísica. 2004, 46–49.

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