ABSTRACT: I perform a numerical calculation of the heat diffusion in a simulated tidally-locked exoplanet. I solve the heat diffusion equation for the surface and the interior of the planet using a finite element grid. I develop a C++ code specifically for this purpose. I take into account both the energy received from the star, the energy radiated into space, the heat source provided by the radioactive decay inside the planet, and the effects of a single-layer atmosphere model. I apply the simulation to the case of Proxima Centauri b. The evolution of the temperature distribution is obtained in the time range from 0 to 15 Gyr. The most important results are the prediction of the possible formation of underground oceans in the dark side and the identification of the habitable zone on the surface of the exoplanet.

KEYWORDS: Physics and Astronomy; Computational Physics; Exoplanets; C++ Simulation; Heat Diffusion.

Introduction

There is a growing number of exoplanets that are being discovered. Many of them are rocky planets, similar to Earth, that orbit in the habitable zone. This is a region around a star in which an exoplanet can host liquid water on its surface and possibly support life.¹² The presence of liquid water is possible only at a narrow range of temperatures. On the Earth’s surface, at atmospheric pressure of 10⁵ Pa, this range is from about 270 to 370 K, but it may vary if the pressure of the exoplanet’s atmosphere is significantly different.³ Understanding the thermal evolution of an exoplanet is fundamental to determining where and when the conditions for habitable zones are satisfied.

The temperature reached by a planet during its evolution depends on several factors: it absorbs a part of the electromagnetic radiation emitted by its star; it produces internal heat with radioactive decay in the planet’s core; it diffuses a part of its heat into space, according to the Stefan-Boltzmann law.²³ A critical parameter is the planetary equilibrium temperature, which is the theoretical temperature that a planet would reach if it were a black body. Thermal equilibrium induced by radiation is reached when the power transmitted by the star to the planet will be equal to that lost by the planet in space. The temperature, \( T_{\text{eq}} \), at which this equilibrium is reached, averaged over the planet’s surface, is equal to:

\[
T_{\text{eq}} = \left[ \frac{L_\ast}{4 \pi D^2 \sigma} \cdot \frac{1 - a}{4 \sigma} \right]^{1/4} \tag{1}
\]

where \( a \) is the albedo, \( L_\ast \) is the star luminosity, \( \sigma \) is the Stefan-Boltzmann constant, and \( D \) is the orbit semi-axis.⁴ The albedo is the fraction of the star radiation reflected into space by clouds, sea-ice, and other bodies on the planet’s surface.

The planet’s atmosphere plays an essential role for two reasons. First, it contributes to raising the average temperature due to the greenhouse effect. Second, it contributes with the winds to transport heat between the illuminated face and the dark one, partly rebalancing the distribution of surface temperature.⁵

I considered a simplified scenario of a single-layer atmosphere model⁶ that I slightly altered to calculate the total energy budget as shown in Figure 1. At the equilibrium,

\[
\sigma T_{\text{eq}}^4 \cdot \epsilon_{\text{atm}} \cdot \sigma T_{\text{atm}}^4 = \epsilon_p T_p^4
\]

\[
\epsilon_p T_p^4 = (1 - \epsilon_{\text{atm}}) \epsilon_p T_{\text{eq}}^4 / (2 - \epsilon_{\text{atm}})
\]

Thus, the temperatures of the atmosphere and the planet are expected to be

\[
T_{\text{atm}} = (2 - \epsilon_{\text{atm}})^{-1/4} T_{\text{eq}}
\]

\[
T_p = \left[ \epsilon_p \frac{1 - \epsilon_{\text{atm}}}{2} \right]^{1/4} T_{\text{eq}}
\]

In fact, from the second equation in Eq. 2, it follows that \( \epsilon_p T_p^4 = 2 T_{\text{atm}}^4 \). Substituting this relation in the first equation, it results in \( T_{\text{atm}} = (2 - \epsilon_{\text{atm}})^{1/4} T_{\text{eq}} \), from which the first of Eq. 3 is derived. By eliminating \( T_{\text{atm}} \) from the first equation in Eq. 2 and rearranging the terms, it follows \( T_{\text{eq}}^{4/4} (1 + \epsilon_{\text{atm}} / (2 - \epsilon_{\text{atm}})) = \epsilon_p T_p^{4/4} / \epsilon_p (2 \epsilon_{\text{atm}}) T_{\text{eq}} \), from which the second equation in Eq. 3 is obtained.

Note that \( T_{\text{eq}} \), as defined in Eq. 1, appears in Eq. 3 only as a reference for the average incoming stellar radiation flux, but in general neither the atmosphere nor the planet’s surface is...
at a temperature $T_{eq}$ at the equilibrium. In the limit $\epsilon_{atm}\rightarrow 1$, the atmosphere temperature could approach the equilibrium temperature $T_{eq}$. However, because of the greenhouse effect, the temperature of the planet’s surface $T_P$ remains always greater than $T_{eq}$ if $\epsilon_{atm}>0$, even if $\epsilon_{atm}\rightarrow 1$.

In this paper, I present a simulation of the heat diffusion for the exoplanet Proxima Centauri b. It orbits around Proxima Centauri, a red dwarf star with a mass of about 12.5% of the Sun’s mass and luminosity of about only 0.1% that of the Sun. Proxima Centauri b is an exoplanet very similar to Earth located at only 0.0485 UA. Proxima Centauri b and its star form an isolated star-planet system subjected only to gravitational tides. Therefore, there is a high possibility that Proxima Centauri b is tidally locked in a nearly circular orbit.⁷

Proxima Centauri b is located in the habitable zone⁸ and, considering an albedo of 0.3, it has an equilibrium temperature of 229 K. Assuming a planet emissivity of 0.9, and an atmosphere emissivity of 0.9, the temperature of the planet is 273 K and the temperature of the atmosphere is 223 K.

## Methods

### Spherical coordinates:

Starting from Fourier’s law of heat conduction, I obtained the heat diffusion equation in spherical coordinates⁹:

$$q = -k \frac{d}{dL}T,$$  \hspace{1cm} (4)

where $q$ is the heat flux in [W] through the surface $A$ in [m²], $k$ is the material conductivity in [W/(m·K)], and $d/L$ is the temperature gradient perpendicular to the surface.

I chose to represent the temperature on a grid of nodes in spherical coordinates. Using Eq. 4, I calculated the heat flux through the faces of an infinitesimal cell centered in $r, \phi, \theta$ (see Figure 2) obtaining the following

$$q_r = -k r^2 \frac{\partial \phi}{\partial r} \sin \theta \cos \phi \frac{\partial T}{\partial r},$$  \hspace{1cm} (5)

$$q_\phi = -k r \frac{\partial r}{\partial \phi} \frac{\partial T}{\partial \phi},$$  \hspace{1cm} (5)

$$q_\theta = -k r \sin \theta \frac{\partial \theta}{\partial \phi} \frac{\partial T}{\partial \theta}.$$

![Figure 2](https://ijhighschoolresearch.org)

Considering a heat capacity $c_p$ and a density $\rho$, from

$$dQ = c_p \rho \frac{dT}{dt},$$

where $V = r^2 \sin \theta dr d\phi d\theta$.

Developing in the Taylor series $q_{r+dr}$ up to the first order

$$q_{r+dr} = q_r + \frac{\partial q_r}{\partial r} dr,$$

The net heat flux in the cell is:

$$q_{cell} = q_{r+dr} - q_r = \frac{\partial q_r}{\partial r} dr.$$

By repeating for all directions and by adding the three contributions, using Eq. 5 and Eq. 7, I solved for the heat diffusion equation in spherical coordinates

$$\frac{\partial}{\partial t} \left( r^2 \sin \theta \frac{\partial T}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( r \sin \theta \frac{\partial T}{\partial \phi} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) = -\alpha \frac{\partial^2 T}{\partial r^2} + Q_{decay},$$

where $\alpha = k c_p \rho$ is the diffusivity constant in [m²/s] and $Q_{decay}$ is the power generated per unit volume by the internal nuclear decays in [W/m³].¹⁰

I further developed the derivatives of the product in $r$ and $\theta$ re-writing:

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = \frac{\partial r^2}{\partial r} \frac{\partial T}{\partial r} + r^2 \frac{\partial^2 T}{\partial r^2},$$

$$\frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) = \sin \theta \frac{\partial T}{\partial \theta} + \sin \theta \frac{\partial^2 T}{\partial \theta^2}.$$

I solved Eq. 10 using a finite element grid. I developed a dedicated C++ code specifically for this purpose. Each cell of the grid is centered in a node at position $(r, \phi, \theta)$ and has a volume of $\Delta V = r \Delta r \sin \theta \Delta \phi \Delta \theta$.

Indeed, to solve Eq. 10 it is necessary to have both first and second derivatives. Therefore, I developed the temperature in the Taylor series up to the second order. For example, let’s consider the radial direction

$$T(r+\Delta r) = T(r) + \frac{\partial T}{\partial r} \Delta r + \frac{\partial^2 T}{2 \partial r^2} \Delta r^2,$$

$$T(r-\Delta r) = T(r) - \frac{\partial T}{\partial r} \Delta r + \frac{\partial^2 T}{2 \partial r^2} \Delta r^2.$$

Adding and subtracting, I found the first and second derivatives

$$\frac{\partial T}{\partial r} = \frac{T(r+\Delta r) - T(r-\Delta r)}{2 \Delta r},$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{T(r+2\Delta r) - 2T(r+\Delta r) + T(r)}{\Delta r^2}.$$

These formulas allow me to compute the derivatives of the temperature for each node of the cell considering the temperatures of the node before and of the node after, along each direction.
Time Evolution:
The evolution of temperature is obtained by discretizing the time in steps of finite duration \( \Delta t \). I first calculated the spatial derivatives at a given time and then I use Euler’s formula to extrapolate the temperature at the next time step

\[
T(t+\Delta t) = T(t) + \frac{\partial T}{\partial t} \Delta t + \ldots
\]  

(15)

where the time derivative at a time \( t \) is provided by Eq. 10.

Euler’s method is known to be unstable, so the time step \( \Delta t \) is chosen adaptively such that at each time the maximum variation of temperature is 10%.

One-Dimension Test:
I tested the code in the one-dimensional case along the planet’s equator. Only heat diffusion was taken into account. The initial condition is \( T_0(\phi)=(300+20\sin\phi)K \). I used a periodic boundary condition since the equator is a closed ring of radius \( r \). In this simple situation, it is possible to derive an analytical solution to Eq. 10: \( T(\phi)=\left(300+20\sin\phi\right)e^{-t/t_c}K \), where the characteristic time is given by \( t_c=r^2/\alpha \). By assuming \( \alpha=2 \ m^2/s \), \( \Delta\phi=0.36 \circ \), and \( r=1000 \ Km \), \( t_c=15 \ Gyr \).

In Figure 3 I show the comparison of the numerical results (red dots) with the analytical solution (blue lines).

Three-Dimension Simulation:
I applied the numerical method described in the section above to the case of the exoplanet Proxima Centauri b. I performed a three-dimension simulation adopting the physical parameters listed in Table 1.

For decay power, density, thermal conductivity, and specific heat capacity I took Earth’s average values.¹¹ For simplicity, I assumed that all physical parameters are uniform in space and constant in time, except for the radioactive decay that decreases with time according to

\[
Q_{\text{decay}}=0.01e^{-t/t_{\text{decay}}}[\mu W/m^2].
\]

(16)

The decay of radioactive isotopes like uranium-235 (235U) contributed a large fraction of radiogenic heat to the early Earth due to its relatively short half-life, \( t_{\text{d}} \). Indeed, for this simulation, I assumed \( t_{\text{d}}=1 \ Gyr \) by considering that for 235U \( t_{\text{d}}=0.7 \ Gyr \) and that \( t_{\text{d}}=t_{\text{d}}\ln(2) \).

In addition, I considered that the radioactive decay is uniformly distributed in the planet’s interior, with no radial dependence. No radioactive decay is considered to be present in the atmosphere.

For the Proxima Centauri planetary system, I considered the parameter values reported in the NASA Exoplanet Archive.¹²

It should be considered that some of the fiducial model parameters used to run the simulation are more uncertain than others. Parameters like, the stellar luminosity, the distance of the planet from the star, and the planet radius are reliable estimates since they came from astronomical observations of the Proxima Centauri system. Others, like the thermal conductivity, are more uncertain because they depend on the composition of the interior of the exoplanet and are not directly accessible. In these cases, average Earth values are assumed. Furthermore, the assumption of constant composition through the entire radius of the planet should be considered as a zeroth-order approximation.

The C++ program used for the simulation is named PLANET CODE and is available in GitHub: https://github.com/GiuliaMurgia03/Simulation-of-Heat-Diffusion-in-a-Tidally-Locked-Exoplanet.

Table 1: Simulation set-up.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma ) crust density</td>
<td>5558 Kg/m³</td>
</tr>
<tr>
<td>( k ) thermal conductivity</td>
<td>5 W/(K·m)</td>
</tr>
<tr>
<td>( t_{\text{decay}} ) initial ( Q_{\text{decay}} )</td>
<td>0.01 ( \mu W/m^2 )</td>
</tr>
<tr>
<td>( \epsilon_p ) specific heat capacity</td>
<td>200 J/(Kg·K)</td>
</tr>
<tr>
<td>( \epsilon_a ) planet emissivity</td>
<td>0.3</td>
</tr>
<tr>
<td>( \epsilon_{\text{atm}} ) atmosphere emissivity</td>
<td>0.9</td>
</tr>
<tr>
<td>( L ) star luminosity</td>
<td>0.00154 ( L_\odot )</td>
</tr>
<tr>
<td>( r ) planet radius</td>
<td>6680 Km</td>
</tr>
<tr>
<td>( \delta ) planet Distance</td>
<td>0.0485 UA</td>
</tr>
<tr>
<td>( \alpha ) planet Albedo</td>
<td>0.3</td>
</tr>
<tr>
<td>( \Delta t ) time step</td>
<td>&lt;0.01 Myr</td>
</tr>
<tr>
<td>( \Delta \phi )</td>
<td>5.6 deg</td>
</tr>
<tr>
<td>( \Delta \theta )</td>
<td>5.6 deg</td>
</tr>
<tr>
<td>Number of grid nodes</td>
<td>16x64x32</td>
</tr>
<tr>
<td>( T_{\text{eq}} ) thermal equilibrium temperature</td>
<td>229 K</td>
</tr>
</tbody>
</table>

Initial conditions:
The simulation begins when the planet has already finished the accretion phase by reaching the final radius. I assumed that the planet has been heated by the release of gravitational energy to a uniform temperature of 1000 K.¹³

Boundary conditions:
I used a periodic boundary condition along the longitudinal direction \( (\phi) \). At the north and south poles of the grid, I assumed that there is no heat flux for \( \theta=0 \) deg and \( \theta=180 \) deg along the latitude direction. Thus, according to Eq. 4, \( \frac{\partial T}{\partial \theta} \theta=0 \). Indeed, from Eq. 14 I obtain \( T(\theta+\Delta\theta)=T(\theta-\Delta\theta) \). I applied the same condition along the radial direction for \( r<0 \) at the center of the planet, obtaining

\[
T(r-\Delta r)=T(r+\Delta r).
\]

For the illuminated side, I considered a radiative boundary condition in which

\[
-k \frac{\partial T}{\partial r} = \epsilon_a \sigma T^4 - 4 A \sigma T^4 \sin \theta \cos \phi - \epsilon_{\text{atm}} \sigma T^4,\]

(17)

I obtained

\[
\frac{\partial T}{\partial r} \left|_r \right. = \frac{\epsilon_a T^4}{k} \sin \theta \cos \phi + \epsilon_{\text{atm}} T^4 (\epsilon_a - \epsilon_p),
\]

(18)
from which I got the radiative boundary condition

$$T(r + \Delta r) = T(r) - \frac{2\Delta r \sigma}{k} \left( 4 \cos \phi \sin \theta T_0 + \epsilon_a T_{\text{rad}} - \epsilon_r T^4 \right).$$

For the dark side, only the radiation from the atmosphere contributes to the heat flux toward the surface and the second term on the right-hand side of Eq.17 is null.

**Results**

I presented the evolution of the temperature for Proxima Centauri b in the time range from 0 to 15 Gyr in Figure 4. The simulation run took about a day of computing time on my laptop. The images were obtained with PLANET CODE using a visualization program written in C++ and OpenGL.

I show the northern hemisphere of the illuminated side of the planet, with a sector removed to illustrate the interior.

![Figure 4](image)

**Figure 4:** Each cell of the grid is colored based on its temperature as represented in the color bar.

At the beginning (t=0 Gyr), the planet is at an average temperature of 1000 K and the surface is starting to cool down due to the radiative losses. After 0.5 Gyr the surface already reaches its equilibrium temperature, while the interior is warming up due to the heat released by nuclear decay.

Note that at the center of the illuminated side, the temperature is between 313-333 K. Around this central hot area, at mid-latitudes, a ring forms with a temperature between 273 and 313 K. Assuming for the exoplanet an atmospheric pressure of $10^5$ Pa, these temperatures permit the existence of liquid water on the surface. Thus, this represents the habitable region of the exoplanet. In the rest of the planet, especially in the presence of moving molten rock. In addition, the simulation reveals a variety of features. The most important is perhaps the prediction of the presence of underground oceans, like those thought to exist beneath the surface of Jupiter's satellite Europa. The fact that Proxima Centauri b is a tidally-locked planet is essential because this creates an asymmetry in the dissipation of internal heat between the two hemispheres of the planet.

The simulation also makes it clear that for a planet in the habitable zone, such as Proxima Centauri b, it is also very important to consider its orbital parameters. In particular, on a tidally-locked planet, only the mid-latitude regions of the illuminated face can be considered habitable. However, life sustained by liquid water may also be possible on the dark side in conjunction with the presence of underground oceans.

In this simulation, the presence of underground oceans appears at about 15 Gyr. This time is larger than the current age of the Universe. However, this is still compatible with the expected lifetime of low-luminosity red dwarfs stars, like Proxima Centauri, which will take millions of years to burn through their fuel.

It cannot be excluded that an ocean below the surface could form earlier if the cooling is more intense because of a lighter atmosphere or a greater distance between the planet and the star.

In the future, I would like to further develop PLANET CODE including developing in the program a physical simulation of the atmosphere and the water bodies.

**Acknowledgments**

I would like to thank Dr. Giambattista Aresu for following me during the project and for his valuable suggestions during the writing process. I would also like to express my gratitude to my parents for their support. Without their encouragement, this idea would not have been possible. I am very grateful to the referees for the useful and constructive comments that helped to improve this paper.

DOI: 10.36838/v4i3.4
References

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