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Enhancements And Applications of Series Acceleration

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ABSTRACT: Data and information are ubiquitous in both nature and human society, playing an integral role in our daily lives. Thus, the way in which we process data/information is of utmost importance as well. This paper aims to enhance the efficiency and accuracy of processing given information/data through the discussion of convergent series. The traditional method of approximating a convergent series is by summing up each term, making it inefficient and troublesome. Here, we devised specific models to face this predicament. Then, they are applied to complex infinite series to test the efficiency. Upon desired outcomes, the model is then placed upon more advanced mathematical simulations and the computation of semiconductor parameters. Finally, we intend to translate this idea into computer algorithms, improving the computation of complex data in future computers and devices.

KEYWORDS: Mathematics; Algebra; Number Theory; fluctuating; monotonic; series.

Introduction

Convergent series can exist in two forms: monotonic and fluctuating. While a monotonic convergent series approaches a certain value steadily from one side, a fluctuating one oscillates between two sides as it approaches a number in the middle.

Methods

Monotonic:

A monotonic convergent series always increases or decreases steadily to a single value.

In order to demonstrate the viability of the new approach for a monotonic series, we must pick an example of which the final sum is known. Then, we will use the newly devised method and experiment with its efficiency against a traditional calculating method. Thus, we can first look at the series

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

When $n \to \infty, S \to \frac{\pi^2}{6}$. We can prove this using

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots = \sum_1^\infty (-1)^{n+1} \left(\frac{x^{2n-1}}{(2n-1)!}\right). \quad [1.1]$$

where x is real number. If we sub $\sqrt{x}$ in the expansion above,

$$\sin\sqrt{x} = \sqrt{x} - \frac{\sqrt{x}^3}{3!} + \frac{\sqrt{x}^5}{5!} - \frac{\sqrt{x}^7}{7!} \ldots = \sum_1^\infty (-1)^{n+1} \left(\frac{\sqrt{x}^{2n-1}}{(2n-1)!}\right), x \geq 0. \ [1.2]$$

and then divide by $\sqrt{x}$ on both sides of the equation, we get

$$\frac{\sin\sqrt{x}}{x} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} \ldots = \sum_0^\infty (-1)^n \left(\frac{x^n}{(2n+1)!}\right). \quad [1.3]$$

Letting [1.3] = 0,

$$\begin{cases} \dfrac{\sin\sqrt{x}}{\sqrt{x}} = 0, [1.4.1] \\ 1 - \dfrac{x}{3!} + \dfrac{x^2}{5!} - \dfrac{x^3}{7!} \ldots = 0, as\ n \to \infty. [1.4.2] \end{cases}$$

As the solutions to [1.4.1] and [1.4.2] are equivalent, both will have n solutions as [1.4.2] is a polynomial with n solutions. Solving for [1.4.1],

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 0, \sin\sqrt{x}, \sqrt{x} \neq 0. \sqrt{x} = k\pi, x = k^2\pi^2, k \in [1,2,3,\ldots,n].$$

Thus, the solutions to [1.4] are

$$1^2\pi^2, 2^2\pi^2, 3^2\pi^2, \ldots, n^2\pi^2.$$

Divide by x^n on both sides of [1.4.2],

$$\frac{1}{x^n} - \frac{\left(\frac{1}{x}\right)^{n-1}}{3!} + \frac{\left(\frac{1}{x}\right)^{n-2}}{5!} - \frac{\left(\frac{1}{x}\right)^{n-3}}{7!} \ldots = 0$$

$$\text{Set } \frac{1}{x} = t, \ t^n - \frac{t^{n-1}}{3!} + \frac{t^{n-2}}{5!} \ldots = 0$$

And we know that

$$\frac{1}{1^2\pi^2} + \frac{1}{2^2\pi^2} + \cdots + \frac{1}{n^2\pi^2} = \frac{1}{6}$$

Thus,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = \frac{\pi^2}{6} \ [1.5]$$

With this,

the sum to infinity of the above series has been proven.

Moving on to speed up the process of calculating a series' sum to infinity, we can first split our methods into three degrees (potentially more).

First degree:

$$S_n = S + \frac{a}{n}, S_{n+1} = S + \frac{a}{n+1}, \begin{cases} nS_n = nS + a[1.6.1] \\ (n+1)S_{n+1} = (n+1)S + a[1.6.2] \end{cases}$$

Let [1.6.2] - [1.6.1], $(n+1) S_{n+1} - nS_n = S$ [1.7].

Here, we only use two terms.

Thus, we have our first approach to potentially calculate the sum to infinity at a faster rate, shown by [1.7].

Second degree:

$$S_n = S + \frac{a}{n} + \frac{b}{n^2}, \ S_{n+1} = S + \frac{a}{n+1} + \frac{b}{(n+1)^2}, S_{n+2} = S + \frac{a}{n+2} + \frac{b}{(n+2)^2}$$

Multiplying both sides by the greatest power of n,

$n^2 S_n = n^2 S + na + b$, $(n+1)^2 S_{n+1} = (n+1)^2 S + (n+1)a + b$, $(n+2)^2 S_{n+2} = (n+2)^2 S + (n+2)a + b$

Here, $an+b$, $a(n+1)+b$, and $a(n+2)+b$ form an arithmetic sequence. $an+b+a(n+2)+b = 2(a(n+1)+b)$

Therefore,

$n^2 S + na + b + (n+2)^2 S + (n+2)a + b - ((n+1)^2 S + (n+1)a + b) = 2S$.

Solving for S,

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$$S = \frac{n^2 S_n - 2(n+1)^2 S_{n+1} + (n+2)^2 S_{n+2}}{2} \text{ [1.8]}$$

Thus, we have the second approach which utilizes the first three terms of the sequence to potentially calculate S_∞.

Third degree: To calculate S from the third degree, we can follow the exact procedures as demonstrated in the first and second degrees.

$$S_n = S + \frac{a}{n} + \frac{b}{n^2} + \frac{c}{n^3}$$

Substituting (n+1),(n+2),and (n+3), we can then solve for S from a set of simultaneous equations.

Ultimately, we can get that

$$S = \frac{n^3 S_n - 3(n+1)^3 S_{n+1} + 3(n+2)^3 S_{n+2} - (n+3) S_{n+3}}{6!}$$

Now, we have an addition of the third approach to speeding up the calculation of infinite series. We can also conclude that the formula for S_∞ $= \frac{\sum_{i=0}^{k}(-1)^i (n+i)^k S_{n+i}}{k!}$, given the degree is k. In these terms, theoretically speaking, the precision of the "shortened" calculation of the sum to infinity of monotonic series can be increased with the increasing number of degrees as shown above.

To show the efficiency of this formula, we can use the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} = \frac{\pi^2}{6} = 1.644934067\ldots$ (Using the first 9 digits for efficiency testing).

First, we can record data calculated by a traditional calculator with a certain number of terms, then use our new method to compare its efficiency to that of the traditional.

Using a calculator:

Table 1: Traditional series calculation with percent error.

n	S_n	Percent Error
1	1	
2	1.25	
3	1.36111111	
4	1.42361111	
5	1.46361111	11.02
6	1.49138888	
7	1.51179705	
8	1.52742205	
9	1.53976773	
10	1.54976771	5.7936
99	1.6348839	
100	1.6349839	0.605
101	1.63508139	
1000	1.643934567	0.0608

As we can see, it took about 1000 terms to get the percent error within hundredths.

Using new approach (Done by substituting terms into the equation):

Here is an example of calculating the first degree:

$T_n = (n+1)\ S_{n+1} - nS_n$

$T_1 = (1+1)S_{1+1} - 1S_1 = 1.5$ % Error = 8.81

$T_2 = 1.5833333$ % Error = 3.74

$T_3 = 1.6111111$ % Error = 2.06

$T_4 = 1.634246031$ % Error = 0.65

As evidenced, the percent error with this approach can equate that of the 100th term of the calculator method within the first 5 terms.

This experimentation method can be repeated using the second order

$$T_n = \frac{n^2 S_n - 2(n+1)^2 S_{n+1} + (n+2)^2 S_{n+2}}{2}:$$

Table 2: Second order acceleration using the derived model with percent error.

n	T_n	Percent Error
1	1.625	1.21
2	1.638888888	0.368
3	1.642361111	0.156
4	1.643611111	0.0804

Then, using the "continuous" properties of this summation calculation, we can calculate the second-order sum S_n (Table 2) with respect to S_n :

$$S_n = \frac{n^2 T_n - 2(n+1)^2 T_{n+1} + (n+2)^2 T_{n+2}}{2}$$

Table 3: Third order acceleration using the derived model with percent error.

n	S_n	Percent Error
1	1.647569444	0.160
2	1.645416649	0.0293
3	1.64493541	0.000653

Using just the first few terms of the original series, we were able to reach a precision of 0.000653%, whilst calculating using a calculator could not achieve this even with thousands of terms. This concludes the efficacy of the monotonic model: around 16,000 times faster using the same or fewer number of terms (Table 3).

Fluctuating:

A fluctuating convergent series is one that shows notable alternation between values but ultimately converges to a definite number.

Like the approach in the monotonic model, we will first pick a convergent series and then test the new method's efficacy on it.

In this section, we will use the series below to demonstrate.

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4} \approx 0.7853981634$$

This is a known convergent series and can be proven via some operations of integration and rearrangements. As we can see, the values alternate between positive and negative as their absolute values get increasingly smaller.

To match the fluctuating series, we may use a fluctuating model as well. This can be done using a geometric model. First, we assume that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = S.$$

$$S_n = S + ab^n, b \in (-1,0)$$

$$\text{When } n \to \infty, S_n \to 0$$

Setting up three equations,

$$S_n = S + ab^n$$
$$S_{n+1} = S + ab^{n+1}$$
$$S_{n+2} = S + ab^{n+2}$$

To cancel out a and b, we may notice that

$$(ab^n)(\ ab^{n+2}) = (ab^{n+1})^2$$

Solving for S,

$$(S_n - S)(\ S_{n+2} - S) = (S_{n+1} - S)^2$$

Thus,

$$S = \frac{S_n S_{n+2} - S_{n+1}^2}{S_n + S_{n+2} - 2S_{n+1}}$$

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Having derived the fluctuating series model, we can now test its efficiency in comparison with that of the traditional method, applying them to the below series:

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots=\frac{\pi}{4}\approx 0.7853981634$ (10th decimal point for precision)

By Calculator:

$$\text{Let } a_n=(-1)^{n+1}*\frac{1}{2n-1}$$

$$\lim_{n\to\infty} S_n=S=\frac{\pi}{4}$$

Table 4: Traditional calculation of series with percent error.

n	S_n	Percent Error
1	1	
2	0.6666667	
3	0.8666667	
4	0.723809538	
5	0.8349286349	6.325
6	0.794011544	
7	0.8209346209	
8	0.7542679543	
9	0.8130914837	
10	0.7604599047	3.175
100	0.7828982259	0.318
101	0.7878733503	
102	0.7829472419	

Applying the fluctuating model,

$$\text{Set } S=\frac{S_nS_{n+2}-S_{n+1}^2}{S_n+S_{n+2}-2S_{n+1}}=T_n$$

Table 5: First terms calculated using the derived model with percent error.

n	T_n	Percent Error
1	0.79166666667	0.798
2	0.7833333333	0.263
3	0.7863095238	0.116
4	0.7849206349	0.0608
5	0.7856782107	0.0357

For even more precision, we can set

$$\frac{T_nT_{n+2}-T_{n+1}^2}{T_n+T_{n+2}-2T_{n+1}}=P_n$$

Table 6: First terms calculated with the derived model with percent error.

n	P_n	Percent Error
1	0.7855263158	0.016
2	0.7853625541	0.00453
3	0.7853410831	0.000727

This exponential increase of accuracy possible because this model only requires three terms to "produce" the new accurate term. Therefore, we can produce an even more accurate term with the three terms from P_n (Table 6)

$$\frac{P_nP_{n+2}-P_{n+1}^2}{P_n+P_{n+2}-2P_{n+1}}=Q_n$$

Plugging the terms in,

Q_1=0.785399835

Percent error= 0.0000213%

Evidently, this acceleration of series is revolutionarily efficient. Requiring only 5 original terms, it can approximate the sum-to-infinity at a precision of 0.0000213% percent error.

Results

The results of the derived models of both fluctuating and monotonic convergent series are promising. With an error of 0.0000213% using just 5 of the original terms, the new fluctuating model has proven to be revolutionarily efficient in the calculation of such a series. The monotonic model showed an error of 0.000653% while using the first four terms of the original series. In short, the data shows that these acceleration models are capable of extremely effective series acceleration (Tables 4-6)

Discussion

Application 1:

The acceleration of convergent series could be applied to many cases, the first being the approximation of many unsolved integrals.

Context:

Lemma 1:

Let $h=b-a$, and $f(x)$ is continuous on the interval $[a,b]$

$$\int_a^b f(x)dx\approx\frac{h}{2}[f(a)+f(b)]+\frac{h^2}{4}\left[f'^{\left(a+\frac{h}{3}\right)}-f'\left(a+\frac{2}{3}h\right)\right]\quad(1)$$

Theorem 1:

Let M=ma x$|f'''(r)|$, the error in (1)

$a\leq R\leq b$

$$|R|=\int_a^b f(x)dx-\left\{\frac{h}{2}[f(a)-f(b)]\right\}-\frac{h^2}{4}\left[f'\left(a+\frac{h}{3}\right)-f'\left(a+\frac{2}{3}h\right)\right]\leq\frac{h^4}{18}M$$

Corollary 1:

Divide the interval [a, b] into n parts.

Let $x_i=a+ih$, i= -,1,2,…,n.

$$x_0=a,x_n=b,h=\frac{b-a}{n}$$

then

$$\int_a^b f(x)dx\approx\frac{h}{2}\left[f(a)+f(b)+2\sum_{i=1}^{n-1}f(x_i)\right]+\frac{h^2}{4}\sum_{i=1}^{n-1}\left[f'\left(x_{i-1}+\frac{h}{3}\right)-f'\left(x_{i-1}+\frac{2}{3}h\right)\right](2)$$

Corollary 2:

The error in (2) =

$$|R_n|\leq\frac{nh^4M}{18}$$

Now, we can use the above to construct a new method to accelerate the approximation of many unsolvable integrals.

Denote $I=\int_a^b f(x)dx$

$$H(h)=\frac{h}{2}[f(a)+f(b)+2\sum_{i=1}^{n-1}f(x_i)]+\frac{h^2}{4}[\sum_{i=1}^{n}f'\left(x_i+\frac{h}{3}\right)-f'(x_{i-1}+\frac{2}{3}h)]$$

$$H\left(\frac{h}{2}\right)=\frac{h}{4}[f(a)+f(b)+2\sum_{i=1}^{2n-1}f(x_i)]+\frac{h^2}{16}[\sum_{i=1}^{2n}f'\left(x_i+\frac{h}{6}\right)-f'(x_{i-1}+\frac{1}{3}h)]$$

$H(h)$and $H\left(\frac{h}{2}\right)$,

m e a n the improved formulas for the calculation of integrals by dividing [a, b] into n and 2n parts respectively. Their errors satisfy

$$|I-H(h)|\leq\frac{nh^4M}{18}=\frac{(b-a)^4M}{18n^3}$$

$$\left|I-H\left(\frac{h}{2}\right)\right|\leq\frac{2n\left(\frac{h}{2}\right)^4M}{18}=\frac{(b-a)^4M}{144n^3}$$

From the above, we can conclude that when n is sufficiently large,

$$\frac{|I-H(h)|}{\left|I-H\left(\frac{h}{2}\right)\right|}\approx 8$$

Then,

$$I - H(h) \approx 8\left[I - H\left(\frac{h}{2}\right)\right]$$

$$I - H(h) \approx -8\left[I - H\left(\frac{h}{2}\right)\right]$$

From this, we can solve for I:

$$I \approx \frac{8}{7} H\left(\frac{h}{2}\right) - \frac{1}{7} H(h)$$

$$I \approx \frac{8}{9} H\left(\frac{h}{2}\right) + \frac{1}{9} H(h)$$

$$I \approx H\left(\frac{h}{2}\right) + \frac{1}{63}\left[H\left(\frac{h}{2}\right) + H(h)\right] = \bar{H}$$

According to the above, we treat 1/63 $[H(h/2)+H(h)]$ as a correction of $H(h/2)$, then obtain $\bar{H}$, a better approximation of I. If we combine the outcome with the model we discussed above, we can conclude a better acceleration method for integrals based on Taylor expansion:

Suppose $S_n = \sum_{k=1}^{n} a_k q_k^n + S$

When $|q_k| < 1, S = S_n, n \to \infty$

A higher rank of the transformation can be generalized as

$$e_{s+1}(S_n) = e_{s-1}(S_{n+1}) + \frac{1}{e_s(S_{n=1}) - e_s(S_n)}$$

S = 1, 2, …

Where $e_0(S_n) = S_n$,

$$e_1(S_n) = \frac{1}{e_0(S_{n+1}) - e_0(S_n)}$$

Set $I = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \ln 2$

Chart 1 shows the outcomes of approximates of I by using the model we previously discussed. After using the transformation 3 times, we reached an accuracy of 10^{-4}, which would have needed $4*10^{-4}$ terms otherwise.

Chart 1:

n	$e_0 = S_n$	e_1	e_2	e_3	e_4	e_5	e_6
0	1	-2	0.7	-102	0.6933	-3852	0.6935
1	0.5	3	0.6905	248	0.6931	12015	
2	0.8333	-4	0.6944	-490	0.6932		
3	0.5833	5	0.6924	852			
4	0.7833	-6	0.6936				
5	0.6167	7					
6	0.7595						

Conclusion

Ultimately, the efficiency of the shank transformation proved to be drastically greater than that of the green function; with more terms used, the accuracy will increase even more exponentially with better efficiency. Fluctuating series acceleration should be paired with the fluctuating model, whilst the monotonic should be paired with the monotonic model, for best precision.

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