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On Geometry of Integers over the Real Plane

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ABSTRACT: Integers are discrete on the real number axis R. In the coordinate plane R, points with integer coordinates $\{(x, y) \in \mathbb{R}^2: x, y \in Z\}$ are also discrete and arranged in a lattice pattern. The study of the properties of integer points in planes and even geometric figures in higher dimensions, or, to put it differently, the study of number theory problems in a geometric way, was pioneered by a paper written by the German mathematician Minkowski in 1891. This branch of mathematics is called the geometry of numbers. In this paper, we use elementary examples to show you what kind of problems the discipline studies, what conclusions are reached, and what techniques are used. We also discuss how the geometry of numbers can be used to study the rational approximation of irrational numbers.

KEYWORDS: Mathematics; Number theory; Geometry of numbers; Lattice point; Rational approximation.

Introduction

In hospital laboratories, technicians drop blood onto a squared glass slide and count the grid points through a microscope to estimate the number of white blood cells. They use the principle that the number of grid points multiplied by the area of the square in a certain graph is approximately equal to the area of the graph. Of course, the graph must not be too strange, for example, there is no integer point in the narrow zone between two lines $x = 3/4$ and $x = 1/4$ in the coordinate plane, but the area of this graph can be arbitrarily large. Further, for a regular graph, the finer the lattice, the smaller the error in the above calculation. Let us analyze this problem mathematically using a very regular graph, which is a circle with the origin (0, 0) as the center and radius r. The equation of a circle in the coordinate plane with the origin at the center and radius r is $x^2 + y^2 = r^2$. The number of integer points in the circle is denoted as N. For each integer point A inside the circle, we draw a square $S(A)$ with A as the center and side length 1 (the sides of the square are parallel to the x and y axes). All these squares are colored to form the graph. Since the area of each square is 1, the area of the colored graph is N. Some of the colored parts go outside the circle, and some of the inside parts are not covered by colored areas. But when the radius of the circle r is large, we can think of the colored area N as an approximation to the area of the circle πr^2. Let us analyze whether this is possible and estimate the degree of approximation as well in the following section.

The Area Estimation:

Each integer point A inside the circle is drawn to be a square S(A) in the above way, and there is no point of S(A) whose distance to A is more than $\sqrt{2}/2$ (which is the distance from A to the vertices of the square), and since A is no more than r from the origin (because A is inside the circle), it follows that each point in S(A) is no more than $r + \sqrt{2}/2$ from the origin, because the sum of the two sides of the triangle is greater than the third side. In other words, the colored figure is all contained in a circle with $r + \sqrt{2}/2$ as radius from the origin. The colored area does not exceed $\pi(r+\sqrt{2}/2)2$. On the other hand, the circle with O as the center and $r - \sqrt{2}/2$ as the radius is all included in the colored area, so the colored area is $N \geqslant \pi(r - \sqrt{2}/2)^2$.

Thus

$$\pi(r^2 - \sqrt{2}r + \tfrac{1}{2}) = \pi(r - \tfrac{\sqrt{2}}{2})^2 \leqslant N \leqslant \pi(r + \tfrac{\sqrt{2}}{2})^2$$
$$= \pi(r^2 + \sqrt{2}r + \tfrac{1}{2})$$

Therefore,

$$|N - \pi r^2| \leqslant (\sqrt{2}r + \tfrac{1}{2})\pi$$

This is an estimate using N as an approximation to πr^2. In fact, this is not the best estimate. Using advanced number theory tools, the error term $\sqrt{2}r + 1/2$ on the right side of this equation can be improved to be even smaller. Dividing both sides of the above equation by r^2, we get

$$\left|\frac{N}{r^2} - \pi\right| \leqslant \left(\frac{\sqrt{2}}{r} + \frac{1}{2r^2}\right)\pi$$

When r is large, the right-hand side error tends to zero more and more, so by choosing a large circle and counting the integer points inside, we can calculate N/r^2 as an approximation of π. We can also take a different view of the equation above by taking the unit circle with the origin at the center and radius of 1, and its area is π. Then we draw horizontal lines $1/r$ apart and vertical lines $1/r$ apart, thus cutting the coordinate plane into small squares with sides $1/r$. Count the lattice points N, since each lattice point corresponds to a small square with area $1/r^2$, so multiplying the number of lattice points N, that is, N/r^2 can be used as an approximation of the unit circle area π, with an error of not more than $\sqrt{2}/r + 1/(2r^2)$. This is what the author did, and we are only giving an estimate here, which is reasonable since the estimate is small when r is large. You may want to do this experiment with a square of paper to see if this is an accurate approximation to π. We can also use this method for other types of figures, as long as they are more regular. For example, the area of an n-sided regular polygon with side length a

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a is $(na^2/4)\tan(\pi(n-2)/2n)$, where $n \geq 3$. You can also calculate the approximation of the tangent function $\tan(\pi(n-2)/2n)$ by counting the lattice points.

3. Existence of Integer Points in Large Areas:

A large and regular planar region must contain a relatively large number of integers. We can ask another question: is there a number A such that any plane figure with area more than a must contain an integer? From the counterexample of the narrow zone given earlier, it is clear that our figure must be more regular. Now we present the first basic result given by Minkowski, the founder of the geometry of numbers. First of all, let's see what he meant by regular regions.

Definition 3.1.:

A plane region is called convex if, for any two points A and B inside the region, the entire line segment joining AB is inside the region.[1]

For example, a figure enclosed by a circle, an ellipse, a regular polygon, and a parallelogram is convex.

Definition 3.2.:

A region in the coordinate plane is said to be symmetric about the origin (0, 0), namely, centrosymmetric, if the point $(x, y) = (a, b)$ is inside the region, then the point $(x, y) = (-a, -b)$ is also inside the region.[1]

A convex region with symmetry about the origin must contain the origin (0, 0), because if we take any point (a, b) of this region, we know by centrosymmetry that $(-a, -b)$ is also in it. Then, by convexity, we know that the line connecting the point (a, b) and the point $(-a, -b)$ is also in it. The origin (0, 0) is the midpoint of this line, so the origin must be in the region. Minkowski says that if the area of such a region is greater than 4, it must contain another whole point besides the origin.[2]

Theorem 3.3.:

Let S be a convex region (enclosed by some curve) symmetric about the origin (0, 0). If the area of S is greater than 4, then the interior of S must contain an integer point other than the origin. If the area of S is equal to 4, then there must be an integer point other than the origin in the interior of S or on the boundary of S.[1]

Proof: We consider all integer points whose coordinates are odd. All horizontal and vertical lines through these integers cut the whole coordinate plane into a lattice, each of which is a square of side 2. The region S is also cut into many pieces, which fall in different squares. Now, by moving each grid by an even number of distances horizontally and vertically, it is possible to reunite it with a square grid with $(x, y) = (\pm 1, \pm 1)$ as its four vertices. The cut region S is then incorporated into this square lattice. Since the area of the square is 4, and the area of S is larger than 4. Therefore, when S is cut and moved over, there must be an overlap. There must be two different points A and B inside S that overlap after being moved over. From the way they are moved, we know that the horizontal x and vertical y coordinates of A and B differ by an even number. Let $A = (a, b)$ and $B = (c, d)$, then $(a - c, b - c) = (2n, 2m)$, where n and m are integers and are not all zeroes.

By the assumption that S is centrosymmetric about the origin (0, 0), it follows from the fact that B is inside S that $-B = (-c, -d)$ is also inside S. Then, from the fact that S is convex, we know that the midpoint of the line segment joining A and $-B$ is

$$\frac{1}{2}(A + (-B)) = \frac{1}{2}(a - c, b - d) = (n, m) \neq (0, 0),$$

which is inside S. It is an integer point not equal to the origin. Now, suppose S is a convex region centered at the origin, but with area equal to 2. By the continuity principle, it follows from the above conclusion that there must be an integer point other than the origin inside S or on the boundary of S. If all integers except the origin are outside the boundary of S, we enlarge S slightly similarly to form the region S', keeping S′ as a convex region with the origin as the center of symmetry, and still keeping the other integers inside S′. But then the area of S is greater than 2, which contradicts the above conclusion. There must be an integer point other than the origin in the interior or on the boundary of S. This completes the proof.

Remark 3.4.:

The requirement in the theorem that the area of S is greater than 4 cannot be improved. Consider, for example, a square with $(\pm 1, \pm 1)$ four points as vertices, which is a convex region symmetric about the origin, with area equal to 4, and no interior integers except the origin (but with eight integers on the boundary).

By taking S to be a variety of convex regions with central symmetry, various interesting conclusions are obtained.

Example 3.5.:

An ellipse with its center at the origin satisfies the conditions of the theorem. If the lengths of the long and short axes of the ellipse are 2a and 2b, respectively, we know that its area is πab. When $\pi ab > 4$, the ellipse must have an integer point other than the origin. If the two axes of the ellipse are the x and y axes, then this result is the same. If $ab > 4/\pi > 1$, then at least one of a and b is greater than 1, so the ellipse must have a point $(0, \pm 1)$ or $(\pm 1, 0)$. But if the ellipse is placed obliquely on the coordinate plane, this result is not ordinary.

More General Results

Let α, β, and γ be positive real numbers, then $\alpha x - y = \gamma$ and $\alpha x - y = -\gamma$ are two parallel lines, and $y = \beta$ and $y = -\beta$ are both parallel to the x-axis. These four lines form a parallel quadrilateral S centered on O, satisfying the conditions of Theorem 3.3. The line between the points $A = (-\gamma, 0)$ and $B = (\gamma, 0)$ is the base of the parallelogram with length 2γ. The height of the parallelogram is 2β. The area of the parallelogram S is $4\beta\gamma$. Thus, when $\beta\gamma > 1$, there exists an integer $(a, b) \neq (0, 0)$ in S. That is, there exist integers a and b that are not all zero such that

$$|\alpha a - b| < \gamma, \ |a| < \beta$$

If we take $0 < \gamma < 1$, $\gamma\beta > 1$, then $a \neq 0$ (since if $a = 0$, then $|b| < \gamma < 1$, thus $b = 0$, which contradicts the assumption $(a, b) \neq (0, 0)$). Since S is symmetric about the origin, $(-a, -b)$ is also in the interior of S. Therefore, it is desirable that a be a positive integer. This proves the following theorem.

Theorem 4.1.:

Let α, β and γ be positive real numbers, and $\gamma < 1$, $\gamma\beta > 1$, then there must be a positive integer a and an integer b such that

$$\alpha < \beta, \quad |\alpha - \frac{b}{a}| < \frac{\gamma}{a}.$$

If we take $\beta = 1/(\gamma)$, we know from the continuity principle that when $0 < \gamma < 1$, there exists a positive integer a and an integer b such that

$$1 \leqslant a \leqslant \frac{1}{\gamma}, \qquad |\alpha - \frac{b}{a}| \leq \frac{\gamma}{a} \leq \frac{1}{a^2}$$

Here γ can be any positive real number less than 1 (not necessarily 1/n), and α can be any positive real number (also a rational number). More generally, let α, β, γ, δ be four real numbers. Let $\Delta = \alpha\delta - \beta\gamma$, then when $\Delta \neq 0$, the lines $\alpha x + \beta y = 0$ and $\gamma x + \delta y = 0$ are two different lines through the origin. Let ε and μ be any two positive real numbers, then the straight lines $\alpha x + \beta y = \varepsilon$ and $\alpha x + \beta y = -\varepsilon$ are parallel, and the straight lines $\gamma x + \delta y = \mu$ and $\gamma x + \delta y = -\mu$ are parallel. These four lines form a parallelogram centered at the origin. Theorem 3.3 is applied. We can prove that the area of this parallelogram is $4\varepsilon\mu/\Delta$. This leads to the following generalized theorem.

Theorem 4.2.:

Let $\alpha, \beta, \gamma, \delta$ be real numbers, $\Delta = \alpha\delta - \beta\gamma \neq 0$, then for every positive real number ε, there exist integers a and b that are not all zero, such that

$$|\alpha a + \beta b| \leqslant \varepsilon, \ |\gamma a + \delta b| \leqslant \frac{\Delta}{\varepsilon}.$$

This is a very precise and powerful result, and we conclude this paper with it.

Conclusion

In this paper, we focus on the geometry of numbers, and apply it to the two-dimensional real number plane after learning the lattice theory proposed by Minkowski. The main theorem we prove can be summarized as follows: if a regular figure around the origin has an area larger than 4, then the region surrounded by it contains at least one lattice point.

References

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