Finding the Exact Value of cos ($\pi/7$)

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ABSTRACT: The values of trigonometry function at $\pi/6$, $\pi/5$, $\pi/4$, and $\pi/3$ have simple root expressions. However, it is not easy to find a record of the value of the trigonometry function $\pi/7$. An article from Wolfram Mathworld states that cos ($\pi/7$) is a solution of a cubic equation without giving a root expression of cos ($\pi/7$). The objectives of this project are to prove that cos ($\pi/7$) is a solution to the cubic equation and to find a root expression cos ($\pi/7$). Further investigation into the cubic equation shows that the other two roots are the values of cos ($3\pi/7$) and cos ($5\pi/7$). Therefore, the scope of the study has been extended to cover the values of cosine at $\pi/7$, $3\pi/7$, and $\pi5/7$. The methods of this study consist of three steps. First, prove that cos ($\pi/7$), ($3\pi/7$), and (5 /7) are solutions to the cubic equation using trigonometric identities. Next, find root expressions of the solutions using Cardano's method. Lastly, identify the root expressions with cos ($\pi/7$), cos ($3\pi/7$), and cos ($5\pi/7$). The result shows that the value of cos ($\pi/7$) can be written in two forms: one with ω and another without ω,

where $\omega = \frac{-1+i\sqrt{3}}{2}$. More explicitly,

$$\cos(\pi/7) = \frac{1}{6}\left(1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re(z\omega)\right) \text{ and } \cos(\pi/7) = \sqrt{\frac{1}{2} - \frac{1}{12}\left(1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re(z)\right)}$$

where $z=\sqrt[3]{1+3\sqrt{3}i}$. Here $z=\sqrt[3]{1+3\sqrt{3}i}$ is defined to be the cube root of $1+3\sqrt{3}i$ that

lies in the first quadrant of the complex plane and *Re(z)* referred to as the real part of the imaginary number *z*. The exact values of *Re(z)*, *Re(zw)*, and *Re(zw²)*, are determined later. Hence, the exact value of cos ($\pi/7$) can be rewritten more precisely as

$$\cos(\pi/7) = \frac{1}{6}\left[1 - 2\sqrt{7}\cos\left(\frac{\arctan(3\sqrt{3}) + 2\pi}{3}\right)\right]$$

and

$$\cos(\pi/7) = \sqrt{\frac{1}{2} - \frac{1}{12}\left(1 - 2\sqrt{7}\cos\left(\frac{1}{3}\arctan(3\sqrt{3})\right)\right)}.$$

KEYWORDS: Mathematics; Algebra; Cubic equation; Cardano's method; Exact value of cosine at $\pi/7$, $3\pi/7$, and $5\pi/7$.

Introduction

The values of trigonometric functions at $\pi/6$, $\pi/5$, $\pi/4$, and $\pi/3$ have simple root expressions. However, it is not easy to find a record of the values of trigonometric functions at $\pi/7$. That makes this problem interesting and challenging. Moreover, the solution to this problem will broaden trigonometric knowledge. An article from Wolfram Mathworld states that cos ($\pi/7$) is a solution of a cubic equation and sin($\pi/7$) is a solution of a sextic equation without giving root expressions of both quantities[1]. It is easier to solve the cubic equation for cos ($\pi/7$) and then use a trigonometric identity $\sin^2(x)+\cos^2(x)=1$ to find sin ($\pi/7$). So this project is aimed to find only the exact value of cos ($\pi/7$).

Quadratic equations are easy to solve by factoring or using the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where the quadratic equation

DOI: 10.36838/v5i6.11

is. However, doing so for cubic equations is not easy at all. Therefore, Cardano's method is introduced in this project to solve the cubic equation.

Several concepts about complex numbers are also introduced in this project. Complex numbers are numbers with a real part and an imaginary part written as $z= a+ bi$ where $a, b \in \mathbb{R}$ and a is the $i = \sqrt{-1}$. real part and b is the imaginary part of the imaginary number z. Imaginary numbers can be expressed in other forms.

$$a + bi = r(\cos\theta + \sin\theta) = re^{i\theta}$$

Where the radius $r = \sqrt{a^2+b^2}$ and θ is the angle that a complex vector makes with the real axis in the complex plane counterclockwise.

In addition, cube roots of unity are constantly used throughout the work. Cube roots of unity are the solutions to the equation x^3-1= 0. They are 1, ω , and ω^2 where $\omega = \frac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \frac{-1+i\sqrt{3}}{2}$. Cube roots of unity have a significant geometric application to this project. When we multiply two imaginary numbers, their lengths are multiplied, and their angles are added. ω, and ω^2 with a radius of one make angles of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ radians respectively with the real axis in the complex plane. So, when we multiply a complex number with ω , and ω^2,the result has the same radius but is rotated by $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ and radians respectively.

Methodos

Firstly, verify that cos (π/7), cos (5π/7), and cos (3π/7) are solutions of the cubic equation $8x^3-4x^2+1= 0$. Then, solve the cubic equation using Cardano's method.[2] After that, identify a root expression by algebraic evaluation with cos (5π/7) . After obtaining the value of cos (5π/7) , use trigonometric identities to derive the value of cos (5π/7) to obtain the values of cos (π /7) and cos (3π/7). Lastly, identify the other two root expressions with cos (π/7) and cos (3π/7) by analyzing in the complex plane

Results and Discussions

Show that cos (π/7), cos (3π/7) , *and* cos (5π/7) *are solutions of the cubic equation* $8x^3-4x^2+1= 0$.

First, define $A=\pi/7$. Get $\pi -3A= 4A$.
Take cosine function to both sides.

$$\cos(\pi-3A)= \cos(4A)$$
$$-\cos(3A)= \cos(4A)$$

Since cos(4A)= cos(2·2A)=2cos^2 (2A)-1, it follows that

$$2\cos^2(2A)+\cos(3A)-1=0.$$

From cos (3A)=4cos^3(A)-3 cos(A), get

$$2\cos^2(2A)+4\cos^3(A)-3\cos(A)-1=0.$$

Using (2A)=2cos^2 (A)-1, obtain

$$2[2\cos^2(A)-1]^2+4\cos^3(A)-3\cos(A)-1=0.$$
$$8\cos^4(A)+4\cos^3(A)-8\cos^2(A)-3\cos(A)+1=0$$
$$(\cos(A)+1)(8\cos^3(A)-4\cos^2(A)-4\cos(A)+1)=0$$

Because $\cos(A)=\cos(\pi/7)\neq -1$, therefore

$$8\cos^3(A)-4\cos^2(A)-4\cos(A)+1=0$$

Or

$$8x^3-4x^2-4x+1=0 \text{ when } x=\cos(\pi/7).$$

Later, assume that B=3 $\pi/7$. Get $3\pi-3B=4B$. .
Take cosine function to both sides.

$$-\cos(3\pi-3B)= \cos(4B)$$

It is the same equation as -cos(A)= cos(4A) in the case of angle $\pi/7$ so it leads to the same equation $8x^3-4x^2-4x+1=0$.

In other words, cos(3π-7) is another answer of the cubic equation. In the same way, cos(5π/7) is one of the answers of the cubic equation as well. Therefore, the answers of the cubic equation are cos(π-7), cos(3π-7), and cos(5π/7) .

Solve the cubic equation $8x^3-4x^2-4x+1=0$ ***by using Cardano's method***

Define $P(x)=8x^3-4x^2-4x+1=0$

Replace x with t+ 1/6, get

$$P(x) = P\left(t+\frac{1}{6}\right) = 8t^3 - \frac{14}{3}t + \frac{7}{27}.$$

Now, know that

$$8t^3 - \frac{14}{3}t + \frac{7}{27} = 0$$

Divided by eight from both sides.

$$t^3 - \frac{7}{12}t + \frac{7}{216} = 0$$

Then, define $t= u+v$, by Cardano's method[2], we know that u^3 and v^3 are roots of a quadratic polynomial $y^2 + \frac{7}{216}y + \frac{343}{46{,}656} = 0.$

Get

$$y = \frac{1}{2}\left(\frac{-7}{216} \pm \sqrt{\left(\frac{7}{216}\right)^2 - 4\left(\frac{343}{46{,}656}\right)}\right)$$
$$= \frac{-7 \pm 21\sqrt{3}i}{432}$$

That is a set $\{u^3, v^3\} = \left\{\frac{-7-21\sqrt{3}i}{432}, \frac{-7+21\sqrt{3}i}{432}\right\}.$

Choose $u^3 = \frac{-7-21\sqrt{3}i}{432}$. Obtain

$$u = \frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i},$$
$$\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\omega,$$
$$\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\omega^2$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$ and define $\sqrt[3]{1+3\sqrt{3}i}$ as a cube root of $1+3\sqrt{3}i$ that lies in the first quadrant of the complex plane.

There are three different cube roots of $1+3\sqrt{3}i$ as shown in Figure 1 below.

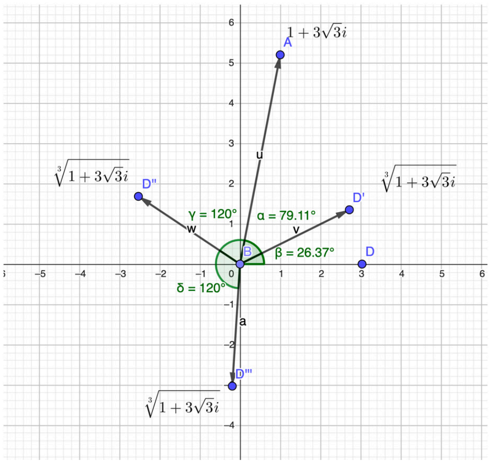

Figure 1: The illustration of $1+3\sqrt{3}i$ and three different cube roots of $1+3\sqrt{3}i$ displayed as $1+3\sqrt{3}i$ in the complex plane where the horizontal axis is the real axis and the vertical axis is the imaginary axis. The figure was created by the author using Geogebra

From Figure 1, the complex vector $1+3\sqrt{3}i$ makes an angle of α= 79.11 degrees with the real axis.

According to the theorem:

If $w = r(\cos\theta + i\sin\theta)$ is a complex number that is not zero, n^{th} root of w has n different roots, which are

$$z=\sqrt[n]{r}\left(cos\left(\frac{\theta+2k\pi}{n}\right)+isin\left(\frac{\theta+2k\pi}{n}\right)\right) \quad where\ k\in\{0,1,\ldots n-1\}.$$

From this theorem, we know that the first cube root of $1+3\sqrt{3}i$ is a complex number that makes an angle of $\beta = \frac{79.11}{3} = 26.37$ degrees with the real axis. Then we can find the next cube root of $1+3\sqrt{3}i$ by rotating the first one by 120 degrees counter-clockwise. The last cube root is determined by rotating the second one by 120 degrees counterclockwise. Now we know that there are three different cube roots of $1+3\sqrt{3}i$. However, for the simplicity of further calculations, we define $\sqrt[3]{1+3\sqrt{3}i}$ to be a cube root of $1+3\sqrt{3}i$ that lies in the first quadrant of the complex plane because both real and imaginary parts of this root are positive.

From $t^3 - \frac{7}{12}t + \frac{7}{216} = 0$ which is in the form of $t^3+pt+q= 0$, know

that $uv = \frac{-p}{3} = \frac{7}{36}$.

That is $v = \frac{7}{36u}$.

Substitute $uv = \frac{-p}{3} = \frac{7}{36}$, obtain

$$v = \frac{7}{36\cdot\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}}$$

$$= \frac{-\sqrt[3]{98}}{6}\cdot\frac{1}{\sqrt[3]{1+3\sqrt{3}i}}\cdot\frac{\sqrt[3]{1-3\sqrt{3}i}}{\sqrt[3]{1-3\sqrt{3}i}}$$

$$= \frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1-3\sqrt{3}i}$$

where $\sqrt[3]{1-3\sqrt{3}i}$ is a conjugate of $\sqrt[3]{1+3\sqrt{3}i}$

From t= u+v,

$$t = \left(\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\right)+\left(\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1-3\sqrt{3}i}\right).$$

From $x = t + \frac{1}{6}$,

$$x = \frac{1}{6}+\left(\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\right)+\left(\frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1-3\sqrt{3}i}\right)$$

$$= \frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}-\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1-3\sqrt{3}i}\right]$$

$$= \frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right].$$

Hence, the first answer to the cubic equation is

$$= \frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right].$$

The steps to arrive at the second and third answers are exactly the same as the steps for the first answer except using different values of u. Hence, the calculations for the second and third answers will be omitted and shown in the appendix.

For $u = \frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\omega$, the second answer is

$$= \frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1+3\sqrt{3}i}\omega+\sqrt[3]{1-3\sqrt{3}i}\omega^2\right)\right]$$

For $u = \frac{-1}{6}\cdot\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\omega^2$, the third answer is

$$= \frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1+3\sqrt{3}i}\omega^2+\sqrt[3]{1-3\sqrt{3}i}\omega\right)\right].$$

Identify the root expression by algebraic evaluation

Consider

$$\sqrt[3]{1+3\sqrt{3}i} = \sqrt[3]{2\sqrt{7}}\cdot\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$$

where $\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$ is the cube root of $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$

that lies in the first quadrant of the complex plane.

DOI: 10.36838/v5i6.11

This is similar to the case above. We define $\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$ to be the cube root of $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$ that lies in the first quadrant of the complex plane for the simplicity of further calculations since both its real and imaginary parts are positive, although we know that there are three different cube roots of $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$.

The radius of $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$ is

$$\sqrt{\left(\frac{1}{2\sqrt{7}}\right)^2+\left(\frac{3\sqrt{3}}{2\sqrt{7}}\right)^2}=\sqrt{\frac{1}{28}+\frac{27}{28}}=\sqrt{1}=1$$

And the radius of $\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$ is the cube root of the radius of $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$. Therefore, the radius of $\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$ is $\sqrt[3]{1}$ which is 1. Then, consider the real parts of $\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$ and $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$. The real part of $\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$ is greater than of $\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$ as shown in Figure 2 where $z_1=\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}$ and $z_2=\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}$.

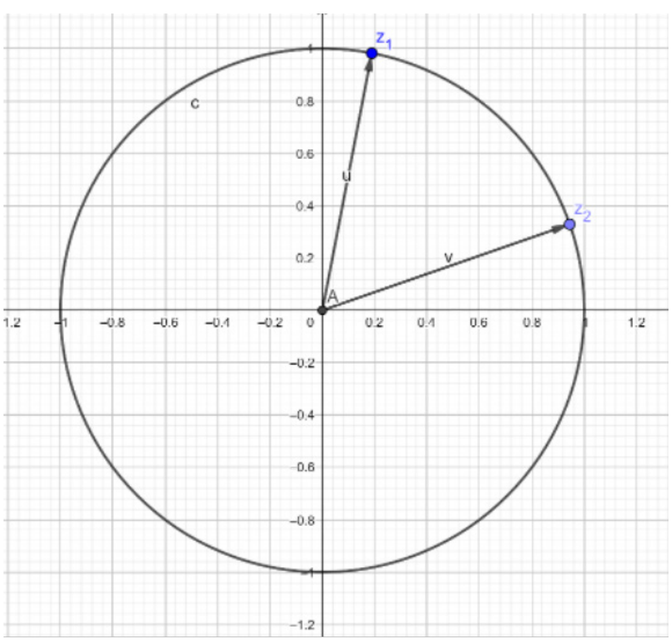

Figure 2: Comparison of the real parts between two complex numbers z_1 and z_2 (not drawn to scale) where the horizontal axis is the real axis and the vertical axis is the imaginary axis. The figure was created by the author using Geogebra.

Therefore,

$$Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)=\sqrt[3]{2\sqrt{7}}\cdot Re\left(\sqrt[3]{\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}}\right)$$
$$>\sqrt[3]{2\sqrt{7}}\cdot Re\left(\frac{1}{2\sqrt{7}}+\frac{3\sqrt{3}i}{2\sqrt{7}}\right)$$
$$=\sqrt[3]{2\sqrt{7}}\cdot\frac{1}{2\sqrt{7}}$$
$$=\frac{1}{2^{\frac{2}{3}}\cdot 7^{\frac{1}{3}}}.$$

Consider the first answer $\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]$,

$$\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]<\frac{1}{6}\left(1-\sqrt[3]{\frac{7}{2}}\cdot 2\cdot\frac{1}{2^{\frac{2}{3}}\cdot 7^{\frac{1}{3}}}\right)$$
$$=0.$$

Since $\cos(5\pi/7)$ is the only value that is less than zero,

$$\cos(5\pi/7)=\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right].$$

Consider $\cos(5\pi/7)=-\cos(2\pi/7)$.
Hence,

$$\cos(2\pi/7)=-\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right].$$

From $\cos\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1+\cos(A)}{2}}$,

$$\cos(\pi/7)=\pm\sqrt{\frac{1}{2}-\frac{1}{12}\left(1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right)}.$$

Because $\cos(\pi/7)>0$,

$$\cos(\pi/7)=\sqrt{\frac{1}{2}-\frac{1}{12}\left(1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right)}.$$

From $\cos(2A)=2\cos^2(A)-1$, $\cos(4\pi/7)=2\cos^2(2\pi/7)-1$.

$$\cos(4\pi/7)=2\left\{-\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]\right\}^2-1$$

$$=\frac{1}{18}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]^2-1$$

From $\cos(3\pi/7)=-\cos(4\pi/7)$,

$$\cos(3\pi/7)=1-\frac{1}{18}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]^2$$

Identify the root expressions by analyzing them on the complex plane:

All three answers from solving the cubic equation are

$$\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]=\cos(5\pi/7),$$

$$\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\,\omega-\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1-3\sqrt{3}i}\,\omega^2\right],$$

and

$$\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1+3\sqrt{3}i}\,\omega^2-\sqrt[3]{\frac{7}{2}}\cdot\sqrt[3]{1-3\sqrt{3}i}\,\omega\right].$$

DOI: 10.36838/v5i6.11

Since we know one of these three answers is the value of , we have only two answers left to identify.

Consider $\sqrt[3]{1+3\sqrt{3}i}$, $\sqrt[3]{1+3\sqrt{3}i\omega}$, and $\sqrt[3]{1+3\sqrt{3}i\omega^2}$ that appear on each answer.

And define that

$$\sqrt[3]{1+3\sqrt{3}i}=x+yi \quad \text{where } x,y\in\mathbb{R}^+$$

because $\sqrt[3]{1+3\sqrt{3}i}$ lies on the first quadrant of the complex plane.

Analyze $\sqrt[3]{1+3\sqrt{3}i}$, $\sqrt[3]{1+3\sqrt{3}i\omega}$, and $\sqrt[3]{1+3\sqrt{3}i\omega^2}$ on the complex plane as shown in Figure 3.

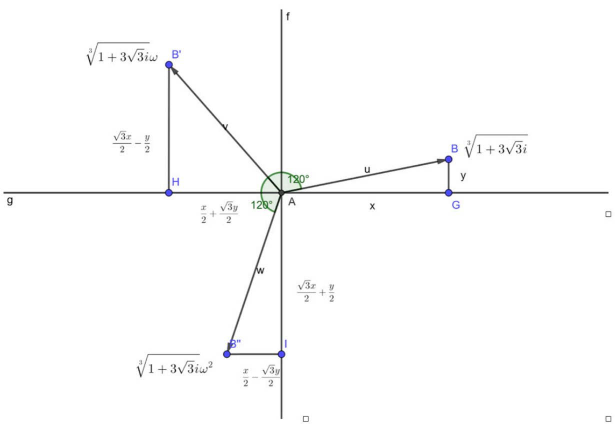

Figure 3: Three complex numbers in the complex plane (not drawn to scale). The figure was created by the author using Geogebra.

Know that

$$\sqrt[3]{1+3\sqrt{3}i\omega}=-\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}\right)+\left(\frac{\sqrt{3}x}{2}-\frac{y}{2}\right)i$$

and

$$\sqrt[3]{1+3\sqrt{3}i\omega^2}=-\left(\frac{x}{2}-\frac{\sqrt{3}y}{2}\right)-\left(\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)i.$$

$\sqrt[3]{1-3\sqrt{3}i}$ is a conjugate of $\sqrt[3]{1+3\sqrt{3}i}$

Therefore,

$$\sqrt[3]{1-3\sqrt{3}i}=x-yi.$$

Now, analyze $\sqrt[3]{1-3\sqrt{3}i}$, $\sqrt[3]{1-3\sqrt{3}i\omega}$, and $\sqrt[3]{1-3\sqrt{3}i\omega^2}$ in the complex plane as shown in Figure 4.

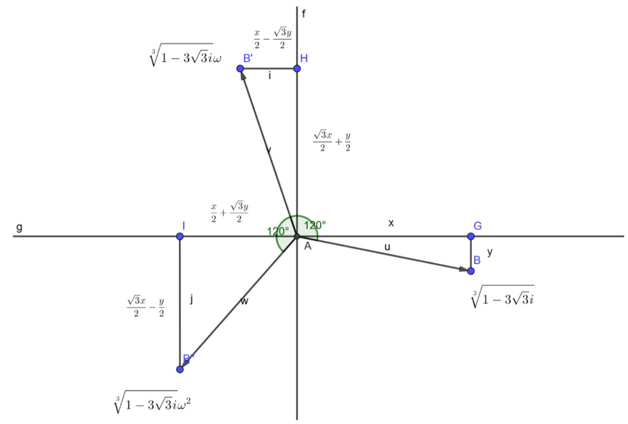

Figure 4: Three complex numbers in the complex plane (not drawn to scale). The figure was created by the author using Geogebra.

Know that

$$\sqrt[3]{1-3\sqrt{3}i\omega}=-\left(\frac{x}{2}-\frac{\sqrt{3}y}{2}\right)+\left(\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)i$$

and

$$\sqrt[3]{1-3\sqrt{3}i\omega^2}=-\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}\right)-\left(\frac{\sqrt{3}x}{2}-\frac{y}{2}\right)i.$$

Since we already know the value of $\cos(5\pi/7)$, we need to analyze only the remaining two answers.

Take a closer look at the second answer.

$$\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1+3\sqrt{3}i\omega}+\sqrt[3]{1-3\sqrt{3}i\omega^2}\right)\right]$$

$$=\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}\right)+\left(\frac{\sqrt{3}x}{2}-\frac{y}{2}\right)i-\left(\frac{x}{2}+\frac{\sqrt{3}y}{2}\right)-\left(\frac{\sqrt{3}x}{2}-\frac{y}{2}\right)i\right]\right\}$$

$$=\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-x-\sqrt{3}y\right]\right\}$$

And take a closer look at the third answer.

$$\frac{1}{6}\left[1-\sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1+3\sqrt{3}i\omega^2}+\sqrt[3]{1-3\sqrt{3}i\omega}\right)\right]$$

$$=\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-\left(\frac{x}{2}-\frac{\sqrt{3}y}{2}\right)-\left(\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)i-\left(\frac{x}{2}-\frac{\sqrt{3}y}{2}\right)+\left(\frac{\sqrt{3}x}{2}+\frac{y}{2}\right)i\right]\right\}$$

$$=\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-x+\sqrt{3}y\right]\right\}$$

Since we define that x and y are positive integers when we define $\sqrt[3]{1+3\sqrt{3}i}=x+yi$ where $x,y\in\mathbb{R}^+$,

$$\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-x-\sqrt{3}y\right]\right\}>\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-x+\sqrt{3}y\right]\right\}$$

and

$$\cos(\pi/7)>\cos(3\pi/7),$$

$$\cos(\pi/7)=\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-x-\sqrt{3}y\right]\right\}$$

$$\cos(3\pi/7)=\frac{1}{6}\left\{1-\sqrt[3]{\frac{7}{2}}\left[-x+\sqrt{3}y\right]\right\}$$

DOI: 10.36838/v5i6.11

Moreover, the answers of $\cos(\pi/7)$ and $\cos(3\pi/7)$ can be written in simpler forms.

$$\cos(\pi/7) = \frac{1}{6}\left\{1 - \sqrt[3]{\frac{7}{2}}\left[-x - \sqrt{3}y\right]\right\}$$

Since $\sqrt[3]{1+3\sqrt{3}i}\,\omega = -\left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right) + \left(\frac{\sqrt{3}x}{2} - \frac{y}{2}\right)$ i and

$Re\left(\sqrt[3]{1+3\sqrt{3}i}\,\omega\right) = -\frac{x}{2} - \frac{\sqrt{3}y}{2}$

$$\cos(\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\,\omega\right)\right]$$

$$\cos(3\pi/7) = \frac{1}{6}\left\{1 - \sqrt[3]{\frac{7}{2}}\left[-x + \sqrt{3}y\right]\right\}$$

Since $\sqrt[3]{1+3\sqrt{3}i}\,\omega^2 = -\left(\frac{x}{2} - \frac{\sqrt{3}y}{2}\right) - \left(\frac{\sqrt{3}x}{2} + \frac{y}{2}\right)$ i and

$Re\left(\sqrt[3]{1+3\sqrt{3}i}\,\omega^2\right) = -\frac{x}{2} + \frac{\sqrt{3}y}{2}$

Therefore,

$$\cos(3\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\,\omega^2\right)\right]$$

Answers in these forms also ensure that the value of $\cos(\pi/7)$ and $\cos(3\pi/7)$ are real numbers as expected as a range of trigonometric functions.

Therefore, we can conclude that

$$\cos(\pi/7) = \sqrt{\frac{1}{2} - \frac{1}{12}\left(1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right)}$$

$$= \sqrt{\frac{1}{2} - \frac{1}{12}\left(1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\,(z)\right)} \qquad (1)$$

$$\cos(\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\,\omega\right)\right]$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\,(z\omega)\right] \qquad (2)$$

$$\cos(3\pi/7) = 1 - \frac{1}{18}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]^2$$

$$= 1 - \frac{1}{18}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\,(z)\right]^2 \qquad (3)$$

$$\cos(3\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\,\omega^2\right)\right]$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(z\omega^2\right)\right] \qquad (4)$$

$$\cos(5\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re\left(\sqrt[3]{1+3\sqrt{3}i}\right)\right]$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2Re(z)\right] \qquad (5)$$

where $z = \sqrt[3]{1+3\sqrt{3}i}$ and $\omega = \frac{-1+i\sqrt{3}}{2}$.

The exact values of $\cos(\pi/7)$ and $\cos(3\pi/7)$ can be written in two forms: one with ω and another without ω. There is only one form for the exact value of $\cos(5\pi/7)$.

Refer to Figure 1 again and write a polar form of $1+3\sqrt{3}i$.

$1+3\sqrt{3}i = \sqrt{28}(\cos\theta + i\sin\theta)$ and $\theta = \arctan(3\sqrt{3})$

Hence,

$$1+3\sqrt{3}i = \sqrt{28}\left(\cos\arctan(3\sqrt{3}) + i\sin\arctan(3\sqrt{3})\right)$$
$$= \sqrt{28}e^{\arctan(3\sqrt{3})i}$$

Recall what we have defined, $\sqrt[3]{1+3\sqrt{3}i}$ is the cube root of $1+3\sqrt{3}i$ that lies in the first quadrant of the complex plane.

So,

$$z = \sqrt[3]{1+3\sqrt{3}i} = 28^{\frac{1}{6}}e^{\frac{\arctan(3\sqrt{3})i}{3}}$$

$$Re(z) = 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right) \qquad (6)$$

$$Re(z\omega) = 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3}) + 2\pi}{3}\right) \qquad (7)$$

$$Re(z\omega^2) = 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3}) + 4\pi}{3}\right) \qquad (8)$$

Therefore, the values of $\cos(\pi/7)$, $\cos(3\pi/7)$, and $\cos(5\pi/7)$ listed in the equation numbers (1)-(5) can be written more precisely by substituting $Re\,(z)$, $Re\,(z\omega)$, and $Re\,(z\omega^2)$ with their values above.

Thus,

$$\cos(\pi/7) = \sqrt{\frac{1}{2} - \frac{1}{12}\left(1 - \sqrt[3]{\frac{7}{2}} \cdot 2 \cdot 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right)\right)}$$

$$= \sqrt{\frac{1}{2} - \frac{1}{12}\left(1 - 2\sqrt{7}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right)\right)} \qquad (9)$$

$$\cos(\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2 \cdot 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3}) + 2\pi}{3}\right)\right]$$

$$= \frac{1}{6}\left[1 - 2\sqrt{7}\cos\left(\frac{\arctan(3\sqrt{3}) + 2\pi}{3}\right)\right] \qquad (10)$$

$$\cos(3\pi/7) = 1 - \frac{1}{18}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2 \cdot 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right)\right]^2$$

$$= 1 - \frac{1}{18}\left[1 - 2\sqrt{7}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right)\right]^2 \qquad (11)$$

$$\cos(3\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2 \cdot 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3}) + 4\pi}{3}\right)\right]$$

$$= \frac{1}{6}\left[1 - 2\sqrt{7}\cos\left(\frac{\arctan(3\sqrt{3}) + 4\pi}{3}\right)\right] \qquad (12)$$

$$\cos(5\pi/7) = \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot 2 \cdot 28^{\frac{1}{6}}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right)\right]$$

$$= \frac{1}{6}\left[1 - 2\sqrt{7}\cos\left(\frac{\arctan(3\sqrt{3})}{3}\right)\right] \qquad (13)$$

Answers to the values of $\cos(\pi/7)$, $\cos(3\pi/7)$, and $\cos(5\pi/7)$ in such forms from above allow us to easily put those expressions into a calculator. Moreover, these expressions reveal relationships among $\cos(\pi/7)$, $\cos(3\pi/7)$, $\cos(5\pi/7)$, and cos

$\left(\frac{\arctan(3\sqrt{3}) + 2\pi k}{3}\right)$ where $k \in \{0,1,2\}$. This could also be future work for me and others to discuss the relationships among those functions.

Conclusion

The exact values of $\cos(\pi/7)$, $\cos(3\pi/7)$, and $\cos(5\pi/7)$ are shown in equation numbers (1)-(5) in two forms: one with ω and another without ω. In equation numbers (1)-(5), those expressions are in terms of $Re(z)$, $Re(z\omega)$, and $Re(z\omega^2)$. Furthermore, since the values of $Re(z)$, $Re(z\omega)$, and $Re(z\omega^2)$ are determined in equation numbers (6)-(8), the exact values of $\cos(\pi/7)$, $\cos(3\pi/7)$, and $\cos(5\pi/7)$ can be rewritten more precisely in equation numbers (9)-(13).

Acknowledgments

I would like to express my very great appreciation to the Institute for the Promotion of Teaching Science and Technology (IPST) and the Development and Promotion of Science and Technology Talents project (DPST) in Thailand for project support and funding. I would also like to offer my special thanks to Dr. Nantawat Udomchatpitak from Mahidol University for his valuable and constructive suggestions and advice during the planning and development of this project.

References

1. Weisstein, Eric W. "Trigonometry Angles–Pi/7." MathWorld – a Wolfram Web Resource. https://mathworld.wolfram.com/TrigonometryAnglesPi7.html. Accessed 24 Feb. 2023.

2. Ratanaprasert, C. พหุนามกำลังสาม Cubic Polynomials. In พีชคณิต [Algebra]. 3rd ed.; The Promotion of Academic Olympiad and Development of Science Education Foundation under the patronage of Her Royal Highness Princess Galyani Vadhana Krom Luang Naradhiwas Rajanagarindra (POSN): Darnsutha Press Co., LTD., 2009; pp 59-65.

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Appendix

Calculations of the second and third answers to the cubic equation

For $u = \frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega}$, get

$$v = \frac{7}{36 \cdot \frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega}}$$

$$= \frac{-\sqrt[3]{98}}{6} \cdot \frac{1}{\sqrt[3]{1 + 3\sqrt{3}i\omega}} \cdot \frac{\sqrt[3]{1 - 3\sqrt{3}i\omega^2}}{\sqrt[3]{1 - 3\sqrt{3}i\omega^2}}$$

$$= \frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega^2}.$$

From $t = u + v$,

$$t = \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega}\right) + \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega^2}\right).$$

From $x = t + \frac{1}{6}$,

$$x = \frac{1}{6} + \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega}\right) + \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega^2}\right)$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega} - \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega^2}\right]$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1 + 3\sqrt{3}i\omega} + \sqrt[3]{1 - 3\sqrt{3}i\omega^2}\right)\right].$$

For $u = \frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega^2}$, get

$$v = \frac{7}{36 \cdot \frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega^2}}$$

$$= \frac{-\sqrt[3]{98}}{6} \cdot \frac{1}{\sqrt[3]{1 + 3\sqrt{3}i\omega^2}} \cdot \frac{\sqrt[3]{1 - 3\sqrt{3}i\omega}}{\sqrt[3]{1 - 3\sqrt{3}i\omega}}$$

$$= \frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega}.$$

From $t = u + v$,

$$t = \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega^2}\right) + \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega}\right).$$

From $x = t + \frac{1}{6}$,

$$x = \frac{1}{6} + \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega^2}\right) + \left(\frac{-1}{6} \cdot \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega}\right)$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 + 3\sqrt{3}i\omega^2} - \sqrt[3]{\frac{7}{2}} \cdot \sqrt[3]{1 - 3\sqrt{3}i\omega}\right]$$

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1 + 3\sqrt{3}i\omega^2} + \sqrt[3]{1 - 3\sqrt{3}i\omega}\right)\right].$$

Thus, the third answer is

$$= \frac{1}{6}\left[1 - \sqrt[3]{\frac{7}{2}}\left(\sqrt[3]{1 + 3\sqrt{3}i\omega^2} + \sqrt[3]{1 - 3\sqrt{3}i\omega}\right)\right].$$

DOI: 10.36838/v5i6.11