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RESEARCH ARTICLE

Comparing the Relativistic Trajectory of a Particle Through a Dipolar and Quadrupolar Magnetic Field

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ABSTRACT: Awareness of the differences in particle motion through dipolar and quadrupolar magnetic fields is vital when understanding what may occur in and around a celestial body. Astronomers need to be able to simulate particles and know what to expect with these different fields. They cannot just look through a telescope and know precisely what is happening. This paper aims to identify and record those differences. Utilizing the Boris algorithm, I will first go through two simple test cases that test the accuracy of the code. These simple test cases include uniform electric and magnetic fields. Then, I will move on to show the dipolar field and then the quadrupolar field. After that, I will compare the two and examine their important differences. These differences show themselves when putting a particle at the same point in both fields with the same initial conditions and plotting their trajectories. Through researching these differences, I found that quadrupolar magnetic fields have a greater effect on particles within them when compared to dipolar magnetic fields. In both cases, the particles follow the magnetic field lines and gyrate around them well. However, the particles are generally confined to a smaller area for quadrupolar fields.

KEYWORDS: Physics and Astronomy; Theoretical, Computational and Quantum Physics; Quadrupolar Magnetic Field; Dipolar Magnetic Field; Boris Algorithm.

Introduction

Astronomers cannot tell what is truly occurring around an object just by looking at an image. That is where particle pusher codes come into play. By using particle pushers, one can accurately simulate how charged particles interact with an object, which can explain or predict various phenomena around celestial bodies. They are vital for understanding what is occurring. The small scale of a particle matters just as much as the large scale of a celestial object. It is especially important to account for relativity in these codes. Particles around massive objects, even smaller massive objects such as the Sun, are accelerated to relativistic speeds.[1] So, without accounting for relativity, the model will be completely wrong.

To code this relativistic trajectory, I used the Boris algorithm.[2] The Boris algorithm is very good at capturing the gyration of a particle around a magnetic field line. Gyration is a main component of a particle's trajectory through a magnetic field. So, this gyration motion is crucial for accurately plotting the trajectory of a particle through magnetic fields, especially complex fields such as a quadrupolar field. The accuracy of my code will be tested with two different analytical solutions by using their particle energy, or Lorentz factor (γ), values: uniform magnetic field and uniform electric field. The Lorentz factor is defined as $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, where v is the velocity and c is the speed of light. The Lorentz factor is being used to test accuracy since it is a key aspect of relativistic motion, and it is quite easy to compare its values obtained analytically and numerically. Precision is key with these complex magnetic fields, so the code must be as accurate as possible.

The two complex fields I will compare are a dipolar magnetic field and a quadrupolar magnetic field. A dipolar field is how many stellar bodies' fields are modeled, such as the Earth and the Sun. However, there are instances where higher pole count models are needed. These instances include but are not limited to star formation and mass accretion.[3,4] Also, with findings from NICER, neutron stars have been found to have multipolar magnetic field components.[5] To determine what is happening in and around a stellar body, we need to differentiate between dipolar and quadrupolar magnetic fields and how they affect a particle within them.

Methods

Every test covered in this paper involves particles moving through electromagnetic fields. For such tests, the relativistic equations of motion are

$$\frac{du}{dt} = \frac{q}{m}(E + v \times B) \quad (1)$$

and

$$\frac{dx}{dt} = v \quad (2)$$

where q is the charge, x the particle position, γ the Lorentz factor, E the electric field, B the magnetic field, and $u = \gamma v$ the vector of relativistic momentum divided by the particle's mass m.

My research was conducted through numerical trajectory experiments using Python. I used the Boris method to compute the trajectories in this research. The Boris method is a very old but well-trusted particle integrator, and the steps about to be shown directly correlate with the steps used in the numerical algorithm. Also, you will notice that the electric field acceleration steps are split into a first half and a second half. This is due to the Boris algorithm using a leap-frog method, essentially meaning that the position of the particle at a midpoint in time is used to push the particle's velocity forward, and then the velocity at the same staggered point in time advances the position.[1] Hence, leap-frog method. The Boris method defines the average velocity in time as

$$\bar{v} = \frac{u^{n+1}+u^{n}}{2\gamma^{n+\frac{1}{2}}} \quad \textbf{(3)}$$

where n represents a discrete computing time step, Δt.

Then there is the inversion step,[6] which is split into 3 separate steps:

- First-half electric field acceleration:

$$u^{-} = u^{n} + \frac{q\Delta t}{2m} E\left(x^{n+\frac{1}{2}}\right) \quad \textbf{(4)}$$

- Rotation step that captures the gyration movement of the particle:

$$u^{+} = u^{-} + \left(u^{-} + (u^{-} \times t)\right) \times s \quad \textbf{(5)}$$

-Finally, there is the second half electric field acceleration:

$$u^{n+1} = u^{+} + \frac{q\Delta t}{2m} E\left(x^{n+\frac{1}{2}}\right) \quad \textbf{(6)}$$

In the rotation step, $t = B\left(x^{n+\frac{1}{2}}\right) q\Delta t/(2m\gamma^{-})$, $s = 2t/(1+t^2)$,

and then $\gamma^{-} = \sqrt{1+(u^{-}/c)^2} = \gamma^{+} = \sqrt{1+(u^{+}/c)^2} = \gamma^{-}$.

As I mentioned in the introduction, there are two analytical cases in which the code has been compared for accuracy by looking at their error in γ. There is the uniform electric field and the uniform magnetic field test cases.

Uniform Electric Field:

A uniform electric field is a field where an electric force accelerates a particle in one direction, and there is no magnetic field (B = 0.) To perform the test, analytical solutions must be derived first. You can derive them from **Equations (1) and (2)**. Solving them, with the initial condition of B=0, you get:[1]

$$x_{an}(t) = \frac{mc^2}{qE}\left(\gamma_{an}(t) - 1\right) \quad \textbf{(7)}$$

and

$$v_{x,an}(t) = \frac{qE}{m}\frac{t}{\gamma_{an}(t)} \quad \textbf{(8)}$$

In the analytical solutions, $\gamma_{an} = \sqrt{1+(qEt)^2/(mc)^2}$.

To set up the test, I initialized a particle at x(t = 0) = (0,0,0) and v = (0,0,0) with the electric field being E = (E_x,0,0). This will cause only the velocity in the x direction to increase uniformly. Both equations had a time step of 0.1 and 0.01 for **Figure 1a** and **Figure 1b**, respectively. The simulation ran the equations 1000 times. As you can see from **Figure 1a** and **Figure 1b**, the error in γ, defined as $\frac{|\gamma - \gamma_{an}|}{\gamma_{an}}$,

is only due to truncation error since the error improves as the time step decreases. So, the error is essentially 0.

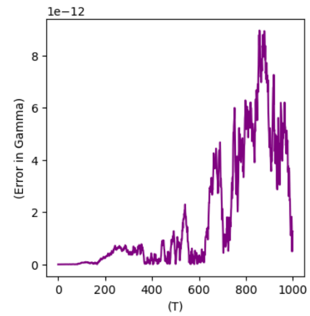

Figure 1a: Relative error in γ for the uniform electric field test with a timestep of 0.1.

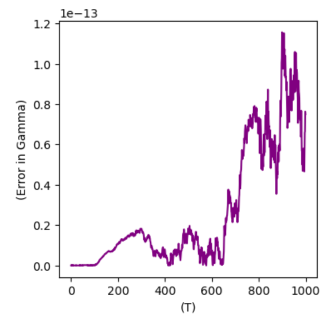

Figure 1b: Relative error in γ for the uniform electric field test with a timestep of 0.1. As you can see, this plot shows a lower error than **Figure 1a**.

Uniform Magnetic Field:

Testing the γ values in a uniform magnetic field is simpler than testing them in a uniform electric field. Because the magnetic field performs no work on the particle, γ will remain constant. Furthermore, the gyro radius will also be constant since γ is constant.[1] You can see this relationship from the equation the gyro radius is derived from:

$$R_c = \frac{\gamma m v_{\perp}}{|q|B} \quad \textbf{(9)}$$

To set this test up, I initialized a particle at x = (0,0,0) and had it start with a velocity v = ($v_{\perp}$,0,0). The magnetic field was set to B = (0,0,B_z) with a time step of 0.1. I then had the code run the equations 10,000 times. As you can see from **Figure 2**, the gyro radius was conserved. So, γ was conserved.

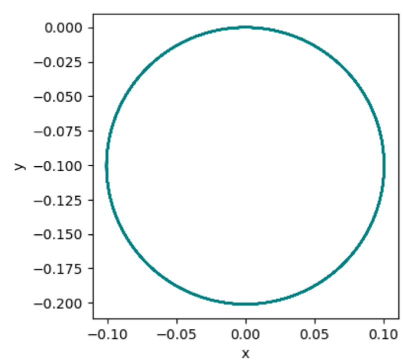

Figure 2: Trajectory in the x – y plane for the uniform magnetic field test. There is practically no change in the gyro radius of the particle. So, γ was conserved, and the code is accurate.

Dipolar Field:

A dipolar field is how quite a few astronomical objects' magnetic fields are modeled, such as the Earth's. Dipolar simply means it has both a north and south pole. A dipole has a very distinct shape consisting of all the field lines converging onto the poles, which also helps see if the code for the dipole is implemented correctly. A dipole can be represented by this equation:[1]

$$B(r,\theta) = \frac{M}{r^3}\left[2cos\theta\hat{r} + \sin(\theta)\hat{\theta}\right] \quad \textbf{(10)}$$

where M is the dipole moment, r is the distance from the dipole's center, θ is the angle from the dipole axis, and its

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orientation in space aligns with the z-coordinate axis. To implement this field into the Boris algorithm, this equation must be converted to the Cartesian coordinate system. Performing this conversion gives you the following.

$$B(x,y,z)=\frac{M}{(x^2+y^2+z^2)^{\frac{5}{2}}}[3zx\hat{x}+3zy\hat{y}+(2z^2-x^2-y^2)\hat{z}] \quad \textbf{(11)}$$

After implementing this into the code, the field lines in the x – z plane look like this:

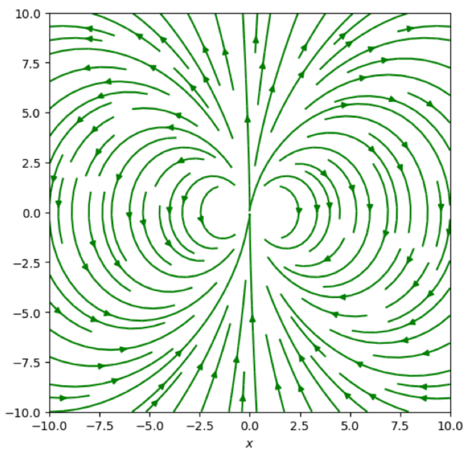

Figure 3: Field lines of a dipole in the x – z plane.

Figure 3 demonstrates the distinct shape I mentioned earlier. To further show off the dipole field, I initialized a particle at x = (1,1,1) with a velocity of v = (0.1,0,0.1) with the dipole moment M = 40 and a time step depending on what I wanted the plots to demonstrate. The equations were performed by the code 10,000 times.

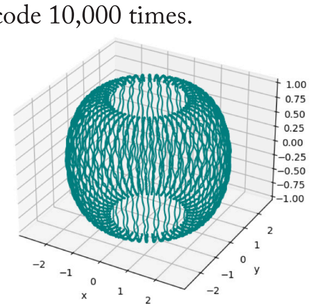

Figure 4a: (Δt=0.385) 3D particle trajectory through the dipolar field. The particle travels around the entire field, gyrating around each field line.

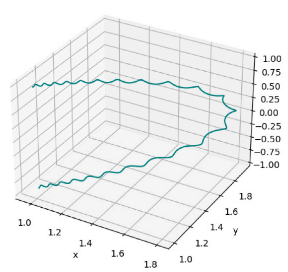

Figure 4b: (Δt=0.003) 3D particle trajectory around one of the dipole field lines. This shows the gyration of the particle clearer than **Figure 4a**. The gyration gets faster as the particle approaches the poles.

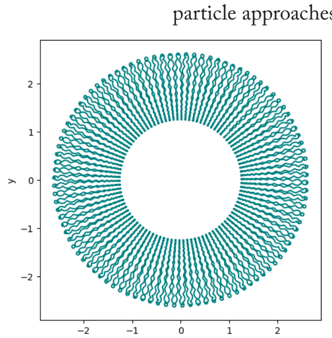

Figure 5: (Δt=0.385) 2D trajectory of a particle through the dipolar field in the x – y plane. This plot further shows the overall movement of the particle in a dipolar field.

As you can see from all three figures **(Figure 4a, Figure 4b, and Figure 5)**, the particle gyrates faster and faster as it approaches either the north or south pole. This reflection effect is an expected behavior. The magnetic force gets stronger the closer the particle is to a pole, which causes the faster gyration. Also, notice how the particle does not stay on the same field line but drifts around the dipole.

Quadrupolar Field:

A quadrupole is a type of multipole containing four poles. These are of increasing interest because several celestial objects are being discovered to have quadrupolar field properties.[3-5, 7] Quadrupoles also have a very distinct shape. They look like a dipole but doubled. A quadrupole can be represented by this equation:[8]

$$B(r,\theta,\phi)=\frac{M}{r^4}\left[-(3\sin^2\theta\cos^2\phi-1)\hat{r}+\sin 2\theta\cos^2\phi\,\hat{\theta}-sin\theta\sin 2\phi\hat{\phi}\right] \quad \textbf{(12)}$$

Where M is the quadrupole moment; r, θ, and ϕ are standard spherical coordinates; and its orientation, just like the dipole, is aligned with the z-axis. Also, like the dipole equation, this must be converted to the Cartesian coordinate system:

$$B(x,y,z)=\frac{M}{r^4}\left(\frac{x\frac{3x^2}{r^2}-1}{r}+\frac{2xz\cos^2\phi cos\theta}{r^2}-2sin\theta\sin^2\phi\, cos\phi\right)\hat{x}$$
$$+\left(\frac{y\frac{3x^2}{r^2}-1}{r}+\frac{2yz\cos^2\phi cos\theta}{r^2}+2sin\theta sin\phi\cos^2\phi\right)\hat{y}+\left(\frac{z\frac{3x^2}{r^2}-1}{r}-\frac{2xsin\theta cos\theta cos\phi}{r}\right)\hat{z} \quad \textbf{(13)}$$

With $r=\sqrt{x^2+y^2+z^2}$, $\theta=$ arccos$\frac{z}{r}$, and $\phi=$ arctan 2 (y,x).

When this equation is implemented, the field lines in the x – y plane look like this:

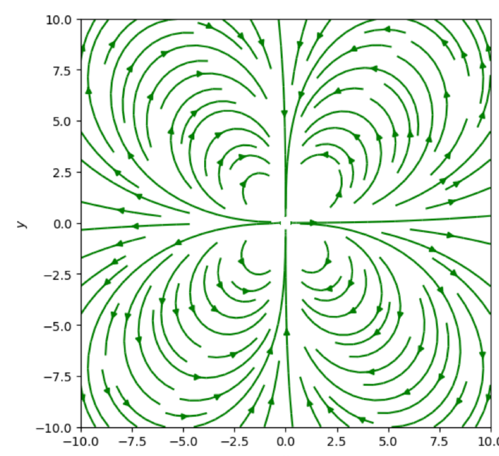

Figure 6: Field lines of a quadrupolar field in the x – y plane.

Figure 6, similar to **Figure 3**, shows the distinct shape of the field. Before I compare, here is another plot showing off the aspects of a quadrupole where x = (1,1,0), v = (0.1,0.1,0.1), quadrupole moment M = 40, and the Δt=0.01. The code also performed the equations 10,000 times.

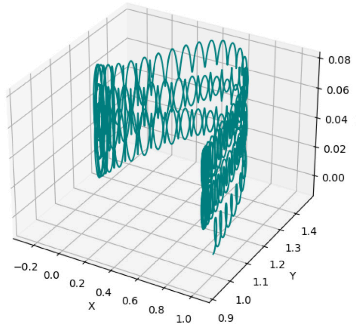

Figure 7: Trajectory of a particle through a quadrupole when initialized at x = (1,1,0) with v = (0.1,0.1,0.1). This plot has an overall similar trajectory similar to the dipole plots **(Figure 4a**, **Figure 4b**, **Figure 5**.**)**

Figure 7 looks fairly similar to the plots of a particle's trajectory through a dipole. It follows the field lines and gyrates along them. The large differences do not show themselves until more extreme conditions are applied.

Results and Discussion

The overall movement of the particle in either the dipolar or quadrupolar field is very similar: the particles follow the field lines and gyrate around them, gyrating faster as they approach one of the poles. Let us look at their differences, though. For all the plots in this section, the Δt=0.01, M (dipole and quadrupole moment) = 40, and equations were performed 10,000 times.

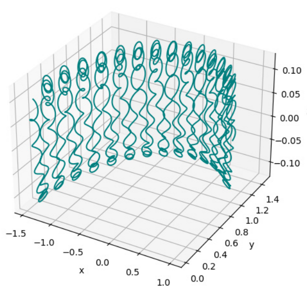

Figure 8a: Trajectory of a particle through a dipole starting at x = (1,1,0) with v = (0.3,0.5,0.1). Notice how the particle is allowed to go through nearly half of the field.

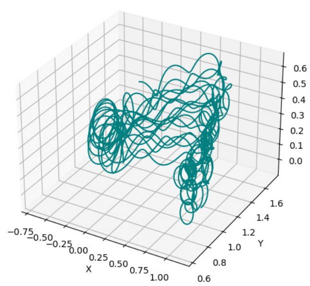

Figure 8b: Trajectory of a particle through a quadrupole starting at x = (1,1,0) with v = (0.3,0.5,0.1). Notice how the particle is contained to a fraction of the entire field.

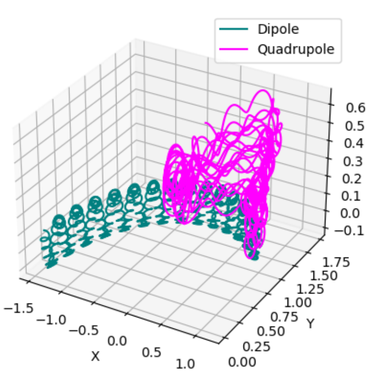

Figure 8c: **Figure 8a** and **Figure 8b** shown on the same plot to aid in discerning their differences.

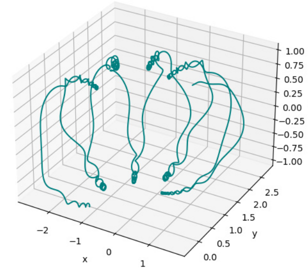

Figure 9a: Trajectory of a particle through a dipole starting at x = (1,1,1) with v = (0.4,0.3,0.1). The trajectory takes a very clean and elegant shape.

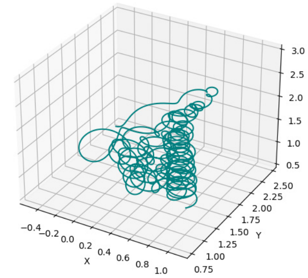

Figure 9b: Trajectory of a particle through a quadrupole starting at x = (1,1,1) with v = (0.4,0.3,0.1). The trajectory is very chaotic.

There are some noticeable differences already in these plots **(Figure 8a, Figure 8b, Figure 8c, Figure 9a, Figure 9b.)** The paths in the dipole plots are much more uniform and elegant. Only two poles act upon the particles, so not too much pushing and pulling occurs. The particles in the quadrupolar field are pushed and pulled by four poles. This causes their trajectories to be more chaotic and random. They still have some general shape, but there are more outliers in the trajectories of the particles in the quadrupolar field than in the dipolar field. Another difference is that the quadrupolar field seems to contain particles in a smaller area when compared with the dipolar field. The plots show how the particle is kept to a much smaller area in the quadrupolar field. The particles in the dipolar field seem like they are on the path to going around the entire field. The particles in the quadrupolar field are kept to a slice of the whole field.

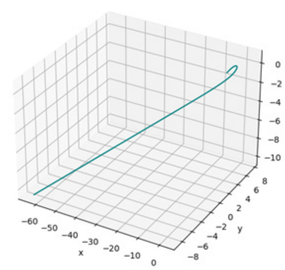

Figure 10a: Trajectory of a particle through a dipole starting at x = (2,3,1) with v = (0.1,0.7,0.1). The particle fully shoots out of the field.

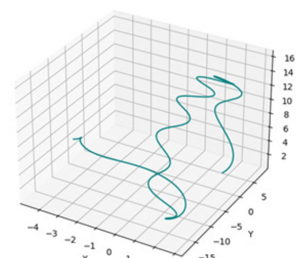

Figure 10b: Trajectory of a particle through a quadrupole starting at x (2,3,1) with v = (0.1,0.7,0.1). The particle seems to drift far from the field, but not fully escape.

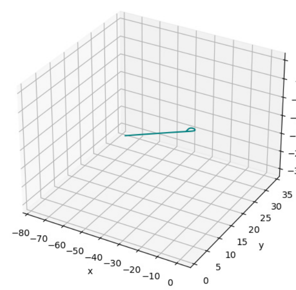

Figure 11a: Trajectory of a particle through a dipole starting at x = (1,1,1) with v = (0.6,0.5,0.5). The particle escapes the field.

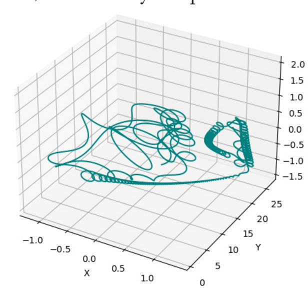

Figure 11b: Trajectory of a particle through a quadrupole starting at x (1,1,1) with v = (0.6,0.5,0.5). There is a lot going on, but it seems that the particle does escape the field. But, it takes more time than it did in **Figure 11a**.

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The differences are quite noticeable again. In all the plots, the high-speed particles escape the field, or they seem to get to the outer reaches of the field. This is because the increase in speed allows the particle to move very quickly to the outer and weaker areas of the field, decreasing the field's effect on the particle and increasing its gyroradius. The increase in gyroradius allows the particle to have a straighter trajectory so that it can escape and get far out in the magnetic field. However, the high-speed particles in the quadrupolar field are contained far better than those in the dipolar field. In **Figure 10a**, the x-axis goes past -60. On the other hand, in **Figure 10b**, the axis that goes the furthest is the y-axis, which only reaches past -15. **Figure 11a** and **Figure 11b** show similar results. The quadrupolar field is not perfect at containing high-speed particles, but it is better than the dipolar field.

Conclusion

Using the Boris method, I compared the relativistic trajectory of a particle through two complex magnetic fields: a dipole and a quadrupole. I first showed what the Boris method is and then tested it with simple test cases: a uniform electric and a uniform magnetic field. Both tests confirmed the code's high accuracy. I then showed what a dipole and a quadrupole look like and briefly described them. After that, I moved on to comparisons.

Through the plots I have shown, it can be deduced that quadrupolar magnetic fields have a more pronounced effect on particles traveling through them when compared to particles traveling through a dipolar magnetic field. This greater effect leads to a more erratic particle trajectory with quadrupolar fields and allows quadrupolar fields to contain high-speed particles better than dipolar fields can. Quadrupolar fields also keep the particles within them in tighter areas than dipolar fields **(Figure 8a, Figure 8b, Figure 9a, Figure 9b.)** All these aspects might affect the radiation from a neutron star with a quadrupolar field to come from a smaller area yet have more complicated trajectories than when compared with radiation from a neutron star with a dipolar field.

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References

1. Bart Ripperda, Fabio Bacchini, Jannis Teunissen, Chun Xia, Oliver Porth, Lorenzo Sironi, Giovanni Lapenta, and Rony Keppens. A comprehensive comparison of relativistic particle integrators. T he Astrophysical Journal Supplement Series, 235(1):21, 2018.
2. P Boris. Relativistic plasma simulation-optimization of a hybrid c ode, paper presented at fourth conference on numerical simulatio n of plasmas, naval res. Lab., Washington, DC, 1970.
3. N. V. Ardeljan, G. S. Bisnovatyi-Kogan, and S. G. Moiseenko. Ma gnetorotational supernovae. Monthly Notices of the Royal Astron omical Society, 359(1):333–344, 05 2005.
4. A G Suvorov and A Melatos. Recycled pulsars with multipolar m agnetospheres from accretion-induced magnetic burial. Monthly Notices of the Royal Astronomical Society, 499(3):3243–3254, 10 2020.
5. Constantinos Kalapotharakos, Zorawar Wadiasingh, Alice K. Har ding, and Demosthenes Kazanas. The multipolar magnetic field o f the millisecond pulsar psr j0030+0451. The Astrophysical Journ al, 907(2):63, Jan 2021.
6. CK Birdsall and AB Langdon. Plasma physics via computer simu lation, Bristol, UK. IOP Publishing, 116(10.1887):0750301171, 1991.
7. J. P'etri. Multipolar electromagnetic fields around neutron stars: e xact vacuum solutions and related properties. Monthly Notices of the Royal Astronomical Society, 450(1):714 742, 04 2015.
8. Samuel E. Gralla, Alexandru Lupsasca, and Alexander Philippov. Inclined pulsar magnetospheres in general relativity: Polar caps fo r the dipole, quadrudipole, and beyond. The Astrophysical Journal , 851(2):137, Dec 2017.

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