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Comprehensive Solutions to Variations of the Knight's Tour Problem

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ABSTRACT: This paper explores solutions for various alterations of the classic Knight's Tour problem, focusing on finding general solutions for certain board classes. The Knight's tour problem involves the movement of a knight from the game of chess, requiring a path to be found for the knight to travel through a grid, landing in every square only once. This paper involves rigorous proofs for standard rectangular boards and through the expansion of these solutions towards non-rectangular 2D boards and to solvable 3D shapes. These solutions are supported by both mathematical proofs and the innovative Knight's Tour Chess Analysis Program (CAP), which allows observation and analysis of knight piece movements on specially designed boards. Through a multidisciplinary approach encompassing graph theory, geometry, algebra, and CAP, this research enhances our understanding of the Knight's Tour problem and its broader implications in computer science and graph theory.

KEYWORDS: Mathematics, Combinatorics, Graph Theory, Game Theory, Knight's Tour, Special Boards Computational solutions to the Knight's Tour Problem.

Introduction

The study of graphs began with the famous Swiss mathematician Leonhard Euler, who needed an efficient method to solve the Königsberg Bridges problem. This problem poses the question of whether it was conceivable to walk around Königsberg once while passing over each of its seven bridges once and then returning to the beginning point.

This was soon solved by Euler through abstracting the landmasses and bridges as vertices and edges. Euler introduced the notion of a graph as a mathematical structure. His graph consisted of a set of vertices and a set of edges, associating each edge with two vertices, which marked the birth of a new branch of mathematics named graph theory. It was observed by Euler that whenever one enters a landmass (vertex) via a bridge (edge), one must leave that landmass (vertex) using another previously untraveled bridge (edge). Euler deduced that for a solution to exist, each vertex must have an even number of edges connecting to it, a condition known as an even degree.[1] This profound insight paved the way for the study of Eulerian paths: a trail that traverses every edge in a graph exactly once while starting and ending at different vertices, establishing a fundamental connection between the existence of Eulerian paths and the condition of even degree for each vertex in a graph.[2]

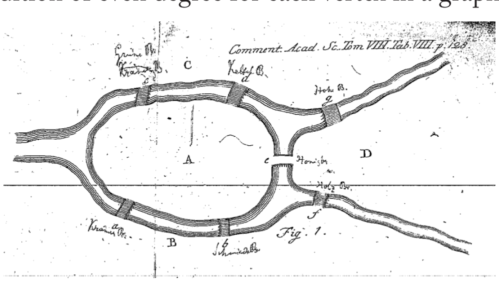

Figure 1: Konigsberg bridge problem.[3]

Building upon this foundation, mathematicians further explored paths that visit each vertex exactly once, known as Hamiltonian paths.[4] Furthermore, Hamiltonian Cycles were explored where the starting vertex and the final vertex of a Hamiltonian path are a single edge apart. This line of investigation was applied to the study of the Knight's Tour problem on the chessboard. A Knight's Tour refers to a sequence of moves executed by a knight where it covers every square on a $n \times m$ chessboard precisely once. This can be done by creating a graph (V,E) where the set of vertices V is denoted by the squares on a chess board and where E is the set of edges between the vertices that are defined by a knight's move. The square in the bottom left of a standard 8×8 chess board will be (1,1) while the bottom right square will be (8,1) and the top right (8,8). A knight's move will be defined as either a horizontal knight's move $(u,v) \rightarrow (u \pm 2, v \pm 1)$, or as $(u,v) \rightarrow (u \pm 1, v \pm 2)$, a vertical knight's move.

This study endeavors to move beyond standard solutions and common board layouts, engaging in new and innovative paths to solve new chessboard configurations through both advanced mathematical discoveries and computational software. This is a valuable area to further research in, as developing these general tools and ideas will allow further research in graph theory and cryptography to advance quicker with a basis to build upon in similar future problems. In the first section of the paper, the methodology and terminology will be extensively discussed, which will be used frequently in the subsequent part of the paper describing the solutions and proofs of extensions to the Knight's Tour problem.

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Methods

To properly investigate the Knight's Tours and Hamiltonian Cycles, we developed a software tool to graphically analyze chess boards quickly and easily, and to present findings in a user-friendly manner. This program is called the Chess Analysis Program (CAP), whose primary function is to find open knight tours. This algorithm incorporates two methods to search for a complete open Knight's Tour and one method for a closed Knight's Tour. The Warnsdorff's heuristic, mentioned below, says to always pick the next knight move that ends on a square with the lowest degree (where 'degree' means the number of possible squares a knight can reach when on a cell). The three algorithms utilized to find Knight's Tours include:

(a) Brute force: This algorithm starts on the specified cell, and then makes a list of possible next cells. It then sorts this list using Warnsdorff's heuristic, and recursively tries every possible knight path until it reaches a solution (hits the safety cap) or has exhausted every possible path, in which case it is guaranteed that no knight's path exists. It contains no heuristics for targets. In practice, the Warnsdorff biassing reduces computational time significantly, although not changing the fundamental Big O time complexity scaling with board size in the general case.

(b) Warnsdorff only (incomplete): This method strictly follows Warnsdorff's rule, restricting the set of possible next knight moves to only those with the lowest degree. As such, this algorithm might miss possible solutions that a complete algorithm would find (see above Brute Force) but will terminate significantly quicker. This algorithm does implement backtracking for equivalently ranked next moves in the case of a tie for Warnsdorff Rank/cell degree.

(c) Hamiltonian cycle: The algorithm implemented was copy-pasted from a reliable source and extensively tested. The completeness of the implemented Hamiltonian cycle algorithm is not required to vindicate the correctness of the theorems presented in this article.

All the algorithms included allow for the manipulation of the standard chessboard's width ($2 \le m \le 20$) and height ($2 \le n \le 20$), as well as the ability to disable any cells (which appear red when disabled). For the brute force and Warnsdorff algorithms, there is also the ability to place targets on which a successful (open) tour must finish. For each cell computed, the status of the computation is placed above the cell: A green tick for success, or a red cross for failure (and a yellow warning in case the computation safety limit was surpassed). For successful tours, green lines representing the exact path a knight can take are also rendered, with red numbers appearing on the top right of the cell the knight lands on in order of movement. Together, this program can create a variety of oddly shaped boards that can be easily visualized, as shown throughout this paper, particularly in the section Special Board Observations Using CAP. The output can also be visually modified, for example, the removal of the sequence numbers for clarity.

The language used was Rust. For brevity, complete reference implementations are not provided in this document since the real implementations involved caching, recursion, compile time inflection, and rather a lot of code in general to enable extremely optimized release binaries (A snippet of high-level logic was presented below).

Some preliminary examples of how the program is used are given below. We utilized this program first in an attempt to locate a Knight's Tour without the usage of disabled cells (colored in red) and target selection (or any square if no targets are specified). For example, to show that a 3 × 3 board has no knight's tours, CAP creates an accurate output as seen in Figure 2, or to show that a 3 × 3 board with the center removed has a Knight's Tour solution starting in the top left, showcasing an output like Figure 4. Some of the time, however, specific solutions are not necessary, and only their existence (or non-existence) is useful. After checking all the other cells in Figure 3, we ended up with Figure 4, demonstrating that a Knight's Tour exists starting on all cells.

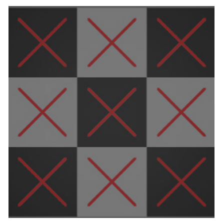

Figure 2: 3 × 3 chess board in CAP showing that no knights tours exist from any starting squares.

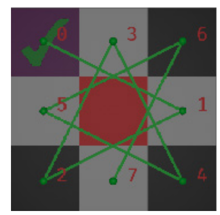

Figure 3: 3 × 3 board with centre removed, showing specific solution starting from top left.

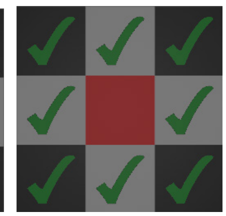

Figure 4: 3 × 3 board with centre removed, not showing specific solution but simply that solutions exist on all outer cells.

Furthermore, for clarity, we define the terms that are commonly used or abbreviated in the following sections of this paper;

(a) Boards: A board is a specific collection of vertices that the knight will be moving on, defined by the dimensions $m \times n$ where m is the number of files and n is the number of rows the board has. The order in which these variables are written does not matter. As graphs are not dependent on any other shapes or positions, rotating a board 90° is trivial and unimportant. This means that 3 × 4 and 4 × 3 boards are equivalent when evaluating general cases, which is important as boards are oriented horizontally for visualization purposes when some may think they should be oriented vertically according to the order of the boards. Note that $u \times v$ will be considered equivalent to $m \times n$, and occasionally the variables m and n will be used as parametric constants.

(b) Degrees: The degree represents the total number of edges connected to a vertex as an endpoint.[5] For example, in Figure 4, the knight piece placed anywhere on the board will have a degree of two.

(c) Connected graph: A graph is connected when every vertex is connected to every other vertex through some path. Therefore, there will always be a path leading through the graph, meaning that any path can be found from any starting point to any ending point.[6]

(d) Hamiltonian cycle/closed knight's tour: A Hamiltonian cycle is a special case of a Hamiltonian path where the final vertex for the Hamiltonian path is within a single move from the first vertex; for example, if a knight were to start on the square (1,1) and finish its tour on the square (3,2) it would be a Hamiltonian cycle, which also known as a 'closed knight's tour'.[7]

(e) Hamiltonian path/open Knight's Tour: A Hamiltonian path is a path that reaches every vertex (or square) once without repetition. Within the context of a Knight's Tour, it is known as an 'open knight's tour'.[8]

(f) Paths: A path refers to a type of graph structure where the vertices are arranged in a specific sequence, and each vertex is directly connected to the next one in the sequence.[9]

(g) Vertical knight's move: A vertical knight's move is where a knight completes the move $(u,v) \rightarrow (u \pm 1, v \pm 2)$, moving two squares vertically and one square horizontally.

(h) Horizontal knights move: A horizontal knight's move is where a knight completes the move $(u,v) \rightarrow (u \pm 2, v \pm 1)$, moving two squares horizontally and one square vertically.

(i) Vertices/cells/squares: Throughout the report, vertices will be interchangeably denoted with the terms 'cells' or 'squares', and when talking about specific vertices/cells/squares the notation (u,v) will be used where u is the file the vertex is on, moving horizontally, while v is the rank of the vertex, moving vertically.

Results and Discussion

Using the methods discussed above, the following section will explore solutions and general observations on an array of problems derived from an extension of the Knight's Tour problem. This paper will explore general observation on n × n rectangular boards, solutions to $u \times v$ boards with a rectangle removed from the center, innovative general solutions for all $3 \times m$, $4 \times m$, $5 \times m$ boards and a brief overview of three-dimensional board layouts.

Open Tours Observations for n × n Boards using CAP:

Paths for $n \times n$ can be easily found when $n \geq 4$, starting from any square. However, the more interesting and valuable question to be proposed is to see to the extent to which an open Knight's Tour can be applied for varying values of n on an $n \times n$ board. It can be observed, using CAP, that when n=2,3,4, all squares are crossed, indicating that no open tours can be found. n=1 is not supported as it is not considered to be a graph due to it not having any edges.

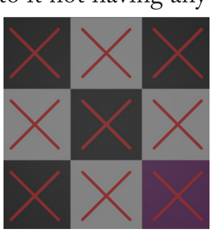

Figure 5: 3 × 3 board in CAP program showing no open tours can be found.

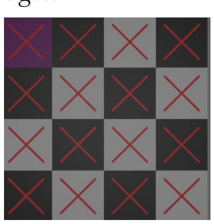

Figure 6: 4 × 4 board in CAP program showing no open tours can be found.

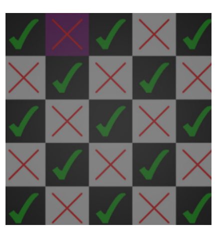

Figure 7: 5 × 5 board in CAP program showing the 'checkboard' pattern

When n=5, the following pattern can be observed where every white cell is unable to complete an open knight tour, while every black cell can. When n=6, the above pattern can be observed where all squares can complete an open Knight's Tour. Repeating ad infinitum, a pattern of boards pertaining to an odd n value forms a 'checkerboard' pattern with white squares crossed and black squares ticked. Where boards pertaining an even n values having all squares ticked, being able to complete an open knight tour in all squares. A continuation of the pattern will be shown in Figures 8 to 11 (Figure 11 shows that the board does not have to be n×n to produce a checkerboard pattern).

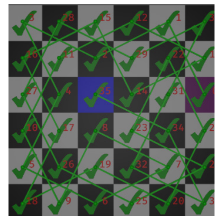

Figure 8: 6 × 6 board open tour.

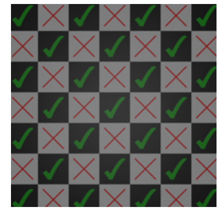

Figure 9: 7 × 7 board forming a 'checkerboard' pattern.

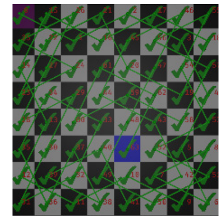

Figure 10: 8 × 8 board allowing open tours on all squares.

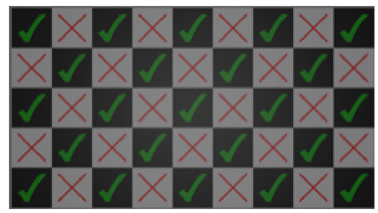

Figure 11: 9 × 5 board forming a 'checkerboard' pattern.

The 'checkerboard' pattern occurs due to the condition that a Hamiltonian path with an odd number of squares has an even number of moves, meaning the path must end on the same color square it started on. Odd-sized boards require a Hamiltonian path to use k squares of one color and k+1 squares of the other remaining color. Therefore, the color that has fewer squares on the board (in these cases above, the white squares) will not have a Hamiltonian path because it cannot end on the color it started on. This applies to all boards with an odd number of squares where one color has k squares and the other has k+1 squares. If there is a difference of more than one square between the two colors, then there would not be any Hamiltonian path. This restriction applies to any 'special boards' (boards that do not follow conventional board shapes). This is shown in Figures 12 and 13, rebalancing a board by removing one square from the k+1 color and unbalancing a board by removing one square from the k color.

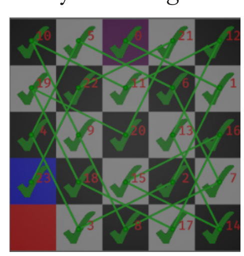

Figure 12: A 5 × 5 odd board with a black square removed.

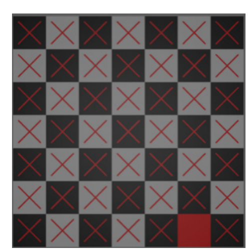

Figure 13: A 7 × 7 odd board with a white square removed.

General Hamiltonian Path Solutions of 3 × m Boards:

Theorem 3. Every board where $m \in \{x \in Z \text{ and } x=4 \text{ or } x \geq 7\}$, has a Hamiltonian path that starts on a corner square and ends

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on the middle square of a rank that is one rank away from an edge of the board:

To start this proof, smaller 3 × m boards will be examined using the CAP solver. It can be proved that 3 × 1, 3 × 2, and 3 × 3 boards do not have a Hamiltonian path, while 3×5 and 3×6 can be seen to not have a solution from the CAP outputs below. 3 × 4 boards have three main solutions, ignoring rotations of the board or directions of the paths. The path that will be utilized in this report is the path of the opposite corner, shown below.

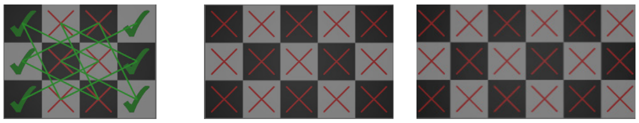

Figure 14: The opposite corner solution is shown for the 3 × 4 board using the CAP solver, while other boards (3 × 5 and 3 × 6) are shown, demonstrating no knights' tours exist from any starting position.

The corner-to-corner solution for the 3×4 board is important as it can be applied to the board compositing method. For example, if two 3×4 boards were placed together to form a 8×3 board then the Hamiltonian path could be used for the first board and then linked to the bottom left corner of the second board i.e., the move (4,3)→(5,1). This technique can be repeated to infinity, proving the solution for every 4k × 3 board where $k \in N$. This technique of adding 3 × 4k boards can be utilized for a variety of boards to prove every 3 × n board has a solution: specifically, the boards 3 × 8, 3 × 9, 3 × 10 and 3 × 7 which can be respectively written as 3 × 4(2),3 × 4(2) +1,3 × 4(2) + 2,3 × 4(1) + 3. Each of these boards has a Hamiltonian path from a corner square to a middle square on the second file to an edge on the opposite side of the board (see Figure 15), which will be referred to as the connector square for 3×n boards. As the path finishes on the middle square and starts on a corner square, it can be used within compositing boards, creating the general solution 3 × 4(1+k) + 3. Using this for all baseboards proves four general equations, 3 × 4(1+k)+3,3 × 4(2+k),3 × 4(2+k)+1,and 3 × 4(2+k)+2, that can be independently linearly scaled to prove solutions to any 3 × n board. Therefore, for every 3 × n board, there exists a Hamiltonian path.

Extending 4×m for Further General Solutions; 4 × m Boards and Hamiltonian Paths:

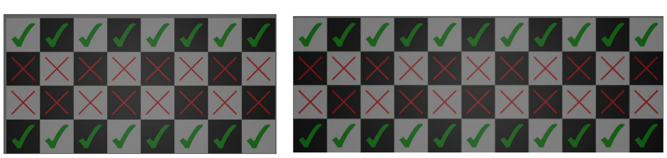

Figure 16: A 4 × 8 and 4 × 11 board showcased in the CAP solver to show the middle ranks do not work as a starting position to a Knight's Tour.

To use the 4×m board for compositing boards, it must start in a lower corner and finish in a square that can reach the bottom corner of the board above it. On a 4 × m board, a vertex that is a finishing point that allows this is the vertex (3,4). This square is three knight moves away from the square (1,1). This is relevant as when connecting two domino sets, the first domino set will finish on the first move (1,1) → (3,2) of the path from (1,1) to (3,4), the second set will start with the second move (3,2) → (5,3) of the path and the end of the second set will be able to finish on the vertex (3,4). This proves 4×m boards can be used as composite boards.

Theorem 3. For every 4 × m board a Hamiltonian path does not exist starting from any cell on ranks two & three.

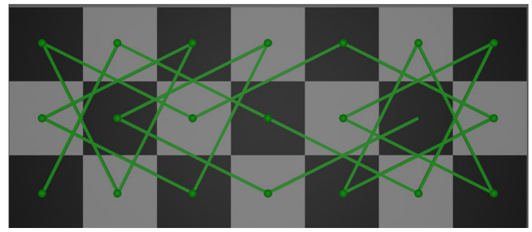

Figure 15: An example 3 × 7 board featuring a Hamiltonian path that started on a corner square and finished on the connector square.

The beginning of this proof was found by C. Flye Sainte-Marie at the 18 April 1877 meeting of the French Mathematical Society, and below shows the explanation and further applications of this proof. To begin with, for every 4×m board, divided into two sets: the top two blocks of the even-numbered files and the bottom two blocks of the odd-numbered files.

This shows that a knight's move from an outer cell of one set can only go to an inner cell of the same set. Therefore, a crossing between two sets can only be done with the inner cells. An open tour has 4m-1 moves on a 4 × m board, and as it is required to have a knight's move on the inner squares between sets, there can be no more than 4m-2 moves linking to an outer square (linking means traveling to or from an outer square). However, there can be no less than 4m-2 moves linking outer squares as there are 2m outer squares, and at least 2m-2 of these squares require two knight moves to be linked to them. The two squares that don't require two moves being the start and the end of the tour, which only require one move each. This means that there must be at least 2(2m-2)+2=4m-2 moves linked to the outer squares.

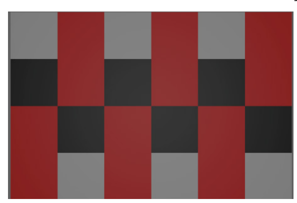

Figure 17: 4 × 6 (M = 6) board split into 'domino' sets. The top two rows and the bottom two rows are considered separate sets in this argument do not work as a starting position to a Knight's Tour.

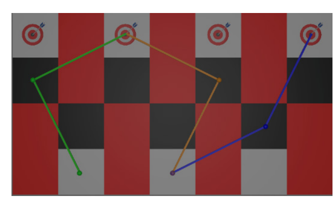

Figure 18: A 4 × 7 (M = 7) board, visually demonstrating that any possible way to cross from the bottom set to the top set requires you to pass through the centre two rows

This result proves that there can be only one crossing between the two sets present in any 4×m board, therefore meaning that each set must have a Hamiltonian path completed with at least one of the ends of the tour being on an inner cell. However, it can be proven that for each set, if an ending of a Hamiltonian path is on an inner cell, the other ending must be on an outer cell. This proof is below. Within one of these sets, what will be referred to as a domino set, every move must alternate between inner and outer cells, which can be seen as all outer cells being

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one color and inner cells being the other color, and every knight move must change colors (Figure 20).

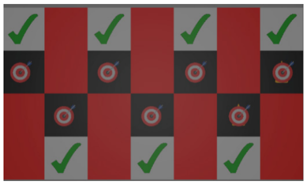

Figure 19: 4 × 7 board, demonstrating using the CAP program that starting on the edges (where the tick marks are) always has a solution (an open Knight's Tour) ending in the centre two rows (where the targets are)

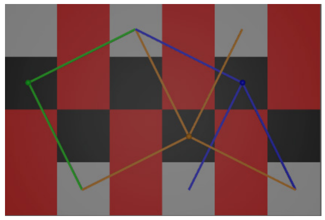

Figure 20: 4 × 6 board split into a domino set, showing that various moves must change colours (e.g. from black -> white and white -> black)

If a knight is on a white square, after $2m$-1 moves, the knight will be on a black square, and vice versa. If a domino set has $2m$ squares, it will have $2m$-1 to move, therefore, the square it starts on must be the opposite color to the square that it finishes on, which means that for a Hamiltonian path within a domino set, one end must be on an outer square, and one end must be on an inner square. This proof shows that the second set the knight enters will have an ending on an outer cell since when entering the new set, the knight will be on an inner cell. Therefore, proving that for every $4\times m$ board there cannot be an ending on the inner ranks.

Domino Sets and Hamiltonian Cycles:

Theorem 5: Every vertex on a domino set can be used as the start or end of a Hamiltonian cycle if the domino set is of size $2m$ where $m \in N$ and m=3 or $m \geq 5$.

Two patterns can be used to establish that every vertex has a Hamiltonian cycle, one for odd-length boards and one for even-length boards. The one for odd boards is rigorously proven, while the solution for even boards is a conjecture with a large basis of evidence and reasoning.

1. Odd Boards

This pattern works with every vertex. To be able to follow the path in Figure 21, start on the lower half of the board, then move in horizontal knight's moves to whichever edge of the board you are closest to while staying on that half of the board. Once at the edge of that half of the board, move vertically to the top half of the board and repeat the process until the other end. Return to the bottom half and complete the whole process again, overall continuing around the board twice. This occurs because after going around the board the first time, the knight will land on the square below the starting vertex, and after completing the vertices the knight missed the first time, it will end one move away from the initial vertex, a Hamiltonian cycle.

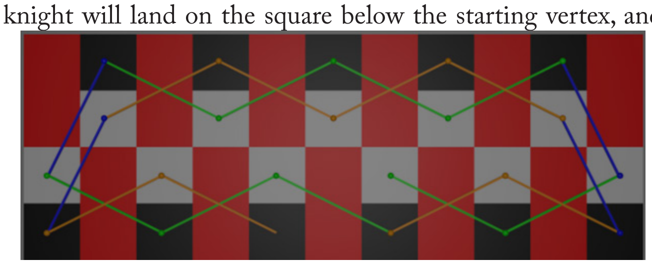

Figure 21: 4 × 11 domino set completed using pattern starting from (7,2) and ending at (5,1) where the green lines are the first time around the board, the orange lines the second time around the board and the blue lines the connecting moves between the two halves of the board.

2. Even Boards

The pattern for boards of even length is similar to the solution for odd boards but is slightly more complex. It involves switching between the top and bottom half of the board every move before getting to four files before the end of the board; before doing either of the following patterns dependent on being on an outer or inner cell when reaching the fourth file from the end, and then returning the other way to repeat the process before going back towards the middle.

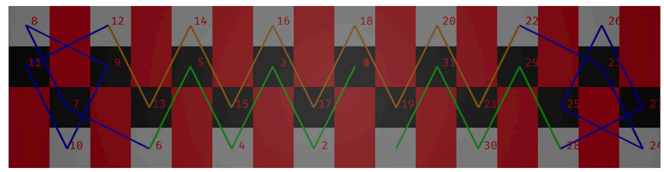

Figure 22: Even domino set completed using pattern starting from (7,2) and ending at (5,1) where the green lines are the first time around the board, the orange lines the second time around the board and the blue lines the moves done in the last 3 files of each side to reach every square on these edges before going back through the middle.

This ends up reaching every square throughout the board, and as the middle of the pattern is repetitive, it can be extended to any length board. Because the solution must go through the entire board twice (think of it as a circle that loops), this means that for every starting square, you can loop back around to it, making a Hamiltonian cycle.

Therefore, by showing Hamiltonian cycles exist for every domino set of size $2m$ where $m \in N$ and m=3 or $m \geq 5$, this shows that $4\times m$ boards have many solutions, including solutions allowing connector boards.

General Hamiltonian Path Solutions of 5 × m Boards

Knight's tours for five by five, six, seven, eight, and nine boards are presented below:

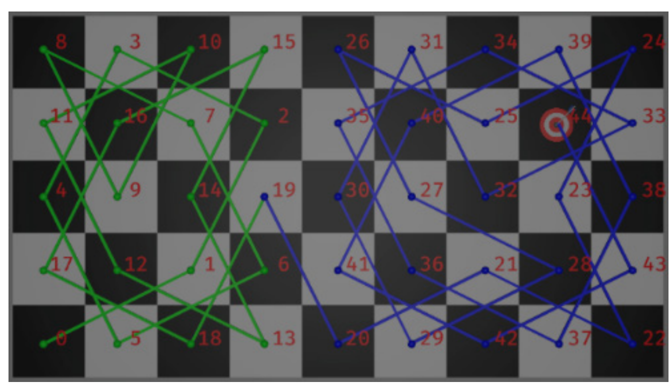

Figure 28: A 5 × 11 board solved using an m = 6 solution (green), a connector move (orange), and an m = 5 solution (flipped along the horizontal so as to align the connector move to the start of the solution). The targets presented are purely visual indicators.

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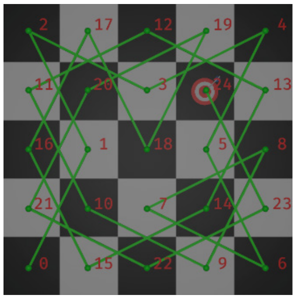

Figure 23: 5 × 5 (M = 5) solution

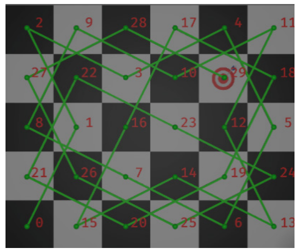

Figure 24: 5 × 6 (M = 6) solution

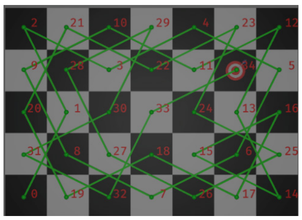

Figure 25: 5 × 7 (M = 7) solution

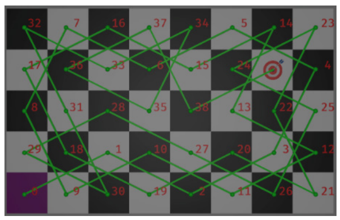

Figure 26: 5 × 8 (M = 8) solution

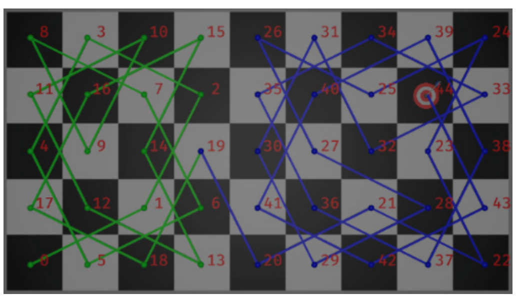

Figure 27: 5 × 9 (M = 9) solution

For Figure 27, the colors are arbitrary but show an interesting separation between the 'left half' solution and the right half solution.

Given these solutions, it can be shown that it is possible to connect solutions to extend a board's width, increasing m by intervals of five, six, seven, eight, and nine (corresponding with known solutions, as presented above). Since the span (over addition) of the set of m values already solved includes all integers greater than five (e.g. 16 = 9 + 7), you can construct a solution for a 5 × m board for all m greater than or equal to five. Hence, a 5 × m board contains a Knight's Tour for m greater than or equal to 5.

Additionally, using CAP it was observed that all even boards where m>5 have a solution on all start locations, and all odd m ≥ 5 solutions formed a checkerboard pattern (with a solution on the bottom left & symmetrically the corners as expected).

The reason that this checkerboard shape happens is because every Hamiltonian path with an odd number of squares has an even number of moves, meaning the path must end on the same color square it started on. Odd-sized boards require the path to use k squares of one color and k+1 squares of the other. Therefore, the color that has fewer squares on the board (in the Figures above the white squares) will not have a Hamiltonian pathway because it cannot end on the color it started on. This applies to all boards with an odd number of squares.

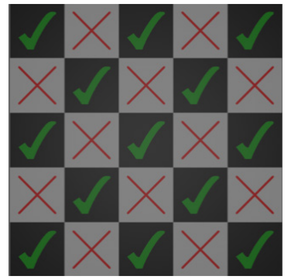

Figure 29: 5 × 5 board in CAP program. The cells with solutions form a checkerboard pattern.

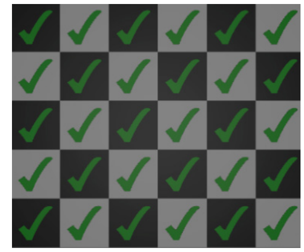

Figure 30: 5 × 6 board in CAP program. Any cells with a tick (all of them) demonstrate a Knight's Tour starting on that square exist.

General Hamiltonian Path Solutions for Every Board:

Theorem 6: A Hamiltonian path exists for every m × n board besides the boards: 1 × m, 2 × m, 3 × (3,5,6) and 4 × 4:

1 × m and 2 × m boards can trivially be seen to have no solutions, while the CAP solver can be used to prove the invalid cases of 3 × m and 4 × m boards.

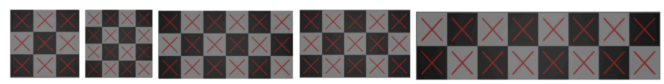

Figure 31: CAP proofs of non-existence of open Knight's Tours on various boards: 1 x m is left as an exercise to the reader, since currently CAP doesn't support m = 1 board sizes.

All viable boards can be written as a composite of smaller boards. Compositing 3 × n boards and 4 × n boards, simply stacking and connecting them on top of each other, allows for the creation of any board. The numbers six, seven, and eight can be written as the sums of threes and fours 3+3,3+4,4+3 respectively. These numbers can be written as 3(2),3(2)+1,and 3(2)+2. Simply adding any a multiple of three, in this case, 3 × n boards generates the equations 3(2+a),3(2+a)+1,3(2+a)+2, where $a \in N$ that represents every possible number greater than five. Adding multiples of three is important as any 3 × n board can be stacked in directions perpendicular to any board that can composite. This means all boards that can be composited can be extended both horizontally and vertically by multiples of three. Combining this with the proof for 3 × m, 4 × m and 5 × m boards prove the theorem 'A Hamiltonian path exists for every m × n board besides the boards: 1 × m,2 × m,3 × (1,2,3,5,6) and 4 × (1,2,4)'.

***Analyzing u × v boards with a (u-2a) × (v-2a) sized rectangles removed from the center*:**

This section focuses on an analysis of chess boards with a rectangular shape removed in the center (marked in red). This section of the paper focuses on analyzing u×v chess boards with a centrally removed (u-4) × (v-4) square. Exploring how the removal of this square divides the board into four rectangles and considers boards with specific dimensions, particularly those with a (u-2a) × (v-2a) removed rectangle

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***Analyzing u × v boards with a (u-4) × (v-4) sized rectangle removed from the center*.**

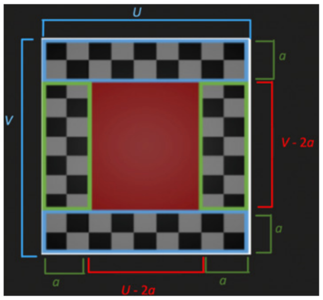

Figure 32: $u \times v$ board with a $(u - 2a) \times (v - 2a)$ rectangle removed, creating 4 separate rectangles, 2 blue rectangles of size $u \times a$ and 2 green rectangles of size $a \times v - 2a$, representing the basis for the proofs used throughout this section.

Firstly, a solution for a $u \times v$ board where a centered rectangle size $(u\text{-}4) \times (v\text{-}4)$ has been removed is investigated, or in other words analyzing boards of size $u \times v$ where $u,v \in \{ x \in N$ where $x \geq 5\}$, and $v \geq u$ with a rectangle of size $(u\text{-}2a)\times(v\text{-}2a)$, where a is the number of rows between the edges of the board and the removed rectangle, for $a \in N$ where $a<u/2 \leq v/2$. A visual evaluation shows that the removal of a rectangle from the center of a board will result in the board being divided into four rectangles: two rectangles of size $2 \times u$ and two rectangles of size $2 \times (v\text{-}4)$. The inner removed rectangle will have the vertices $(1+a,1+a),(1+a,v\text{-}a),(u\text{-}a,1+a),(u\text{-}a,v\text{-}a)$.

Removing a $(u\text{-}2a) \times (v\text{-}2a)$ sized rectangle from the center of a board will divide it into four rectangles, two rectangles of size $u \times a$ (the blue rectangles in Figure 32) and two rectangles of size $(v\text{-}2a) \times a$ (the green rectangles in Figure 32). The proof for any subdivided rectangles that are of a size greater than $u \times 3$ can be solved using the general solution for rectangular boards found earlier, meaning this section will be divided into three parts: finding non-Hamiltonian paths for any board with a rectangle removed, finding Hamiltonian paths when a=2, and finding paths when $a \geq 3$. It is important to note that for both sections below, the notation $u \times v *a$ indicates the size of the board and the size of the rectangle removed from the center of it.

Paths for any Board with a Rectangle Removed:

For a path to exist on any board, the board must be a connected board. In graph theory, a graph is connected when there exists a path (one or more continuous edges) from every vertex to every other vertex, meaning that there always exists some set of edges that can connect any two vertices. From this, it can be observed that starting at any vertex, we can reach any other vertex on a graph. As an application of these concepts to the Knight's Tour problem, it can be understood that any board where all vertices can be represented as a connected graph will have a path.

The inherent complexities of the Knight's Tour are exemplified by the fact that a king being able to move around a chessboard freely does not make it connected. There exist infinitely many variations of possible boards with removed squares. There can be no definite proof regarding the eligibility of boards, besides the element of attempting to represent them as a connected graph. In other words, we will restrict our research to removed rectangles of the form containing removed width $=u\text{-}2a$ and removed height $=v\text{-}2a$ from the center of the board.

For rectangular boards, we have proven that for any $m \times n$ board where $m,n \geq 3$, there exists a solution besides the 3 × (3,5 or 6) board. The reason for this can be seen in the fact that it is not considered a connected graph, with the square at the center of the board being allocated with a degree of 0, meaning it has no connecting squares. Using this, it can be stated that any $u \times v$*a where $a \geq 3$ has a path as they comprise m × n rectangles that can connect with each other.

The reason $2 \times m$ boards do not have paths is that they can be defined as four different connected graphs, as each square is of degree two and cannot connect to other sets.

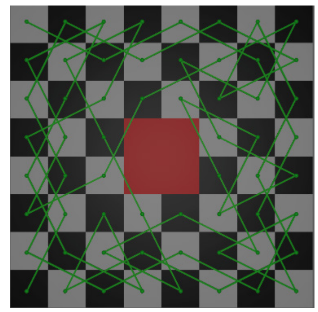

Figure 33: A specific example of a 8 × 8 * 3 board with an open tour.

Figure 34: a 2 × 8 board, coloured in such a way as to demonstrate the four distinct connected graphs (4 different colours)

However, for every board-sized $u \times v$*2 there is a solution, but the method is different from the one given before; where if four cells in a square shape can be proven to be able to all reach each other. On a $u \times v$*2 board there will be a path around the board. This is because the individual rectangles that the $u \times v$*2 boards are divided into all contain four individual connected graphs that can be passed through one at a time.

If every vertex at one of the rectangles can be reached, then all four graphs can be connected, and therefore, that $2 \times u$ section of the board can be passed through. Therefore, starting in a 4-celled square and reaching every cell within that square proves that a path exists around every $u \times v$*2 board. It can also be proven that a 4-celled square in the corner of a $u \times v$*2 can have every cell reach every other cell within that square. From this, it can clearly be seen that a path exists around every $u \times v$*2 board.

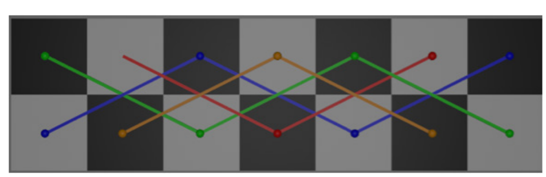

Figure 36: 2 × 7 board with different colours to show different paths around the board.

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Combining this with the statement that there exists a path around every $u \times v$*a board, where $a \geq 3$, shows there is a path for every board of size $u \times v$*a where $a \geq 2$. This is only one specific case of a special board, and further explorations into a variety of special board patterns could be conducted, but this paper will not examine any more arbitrarily chosen special patterns, instead now focusing on Hamiltonian paths for $u \times v$*a boards.

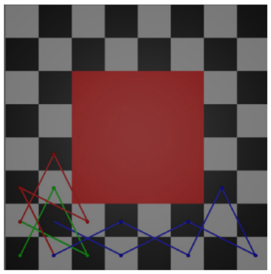


Figure 35: 8× 8 * 2 board where it is shown every cell in a 4-celled corner square can be reached from any other cell in the corner square, with the path from each cell in the corner square to the other represented in a different colour.

Analyzing General Hamiltonian Path Solutions For u × v*2 Boards:

Theorem 1. A Hamiltonian path exists on every board of size $u \times v$*2 , where u=4k+1 and v=2l+1 or u=2k and v=2j+1 where $k,l,j \in N$,where $j \neq 1,2$ and $l \neq 1$:

This theorem uses induction to prove the general case stated above. Removing a rectangle of the size (u-4) × (v-4) from a $u \times v$ board will result in various 2 × u rectangles, as can be seen in Figure 32. While a singular 2 × n rectangle does not have a solution, these four connected rectangles can have a Hamiltonian solution.

Examining the properties of 2 × u boards shows the basis for this proof. Each color shows a self-contained path, a set that cannot connect to any other set. When each path is combined, every vertex is visited exactly once. Because the $u \times v$*2 board has four 2 × m rectangles arranged in a quasi-circular motion, there will be some boards where the four sets can be completed through four 'circuits' around a board. A circuit must reach every square in a set and then switch over to another set for another circuit. Looking at Figure 36, shows that at any point on the board, there will be a square of four cells where each cell is part of a different set. As this happens at any point of a 2×m board, what this means is that the four sets together form every square of the board in these rectangles.

The solution for this section relies on two main sections of these boards: Firstly, the corner squares, groupings of four cells in each corner of the board that will be in the same form on every single board and can therefore be used as part of a general pattern. These corner squares' bordering cells that have degrees greater than two, giving them the ability to switch between sets. If a Hamiltonian path starts in the bottom left corner square, all circuits will conclude and end in that bottom left corner square.

Secondly, the inner pathways, the cells in between each corner square, have a simple structure and can be extended by a certain length. These squares will all have a degree of two, meaning that they are simplistic in their possible paths, only having one direction/square to move to after they have been moved into (See Figure 38).

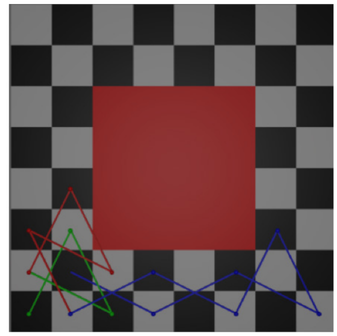


Figure 37: 8 × 8 * 2 board showing two specific cells with a degree greater than two (allowing for set switching).

Because these squares are essentially 'passing squares', the only importance of these squares is how many there are, as they just act as intermediary squares between the corners of the board. Therefore, proving they can be extended will allow existing solutions for $u \times v$*2 to be extended, proving the existence of general solutions. These pathways can be extended by four squares. The reason for this can be seen in how horizontal knight moves work (Figure 38). Two horizontal knight's moves on a 2×m board will create a simple pattern, where if a knight starts on the square (a,b) it will have the following move pattern:

$(a,b) \rightarrow (a \pm 2, b \pm 1) \rightarrow (a \pm 4, b)$

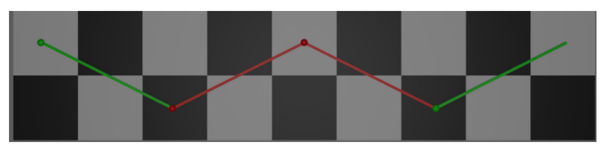
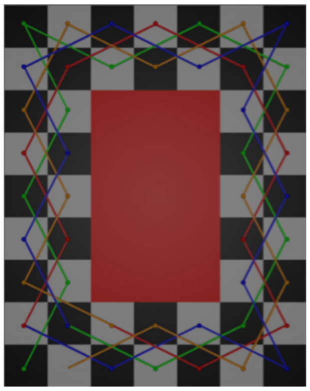


Figure 38: a 2 × 9 board with a move starting from square (1,2) and finishing on square (9,2). The red lines indicate how the original two moves were extended by four columns by adding in an intermediate two moves that extended the path by two squares.

This indicates that a knight can move from square (a,b) to square ($a \pm 4$,b). This notion is a vital component of this proof as lengthening a board by four squares will not affect the final location of a set. From what was explained before about the four different sets as four cells in a square, the entirety of the four sets will not be impacted if an extra four rows are added. Below, Figure 39 shows both a 5 × 7 board and a 9 × 7 board divided into four circuits with each different color representing its own circuit. The corner squares have the same arrangement of colors, indicating no change has occurred due to this extension method. Because the arrangement of sets in the corner squares is the basis of any $u \times v$*2 board, therefore extending the board in any direction by rows will provide a Hamiltonian path.

Therefore, if a solution exists on a certain board $u \times v$*2, then a solution exists on the board (u+4a) × (v+4b)*2. Using this basis, a variety of boards can be proved. A gallery listed below will show examples of boards where u=4k+1 and v=2l+1,or u=2k and v=2j+1 where $k,l,j \in N$,where $j \neq 1,2$ and $l \neq 1$. Using this and the notion that every board can be extended by four squares in any direction proves the theorem stated in "Extending 4×m for Further General Solutions; 4×m Boards and Hamiltonian Paths".



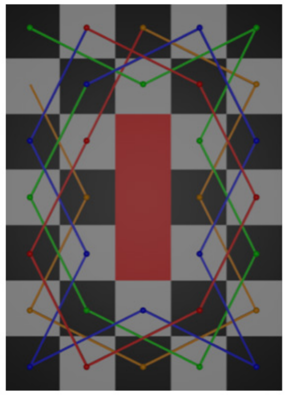
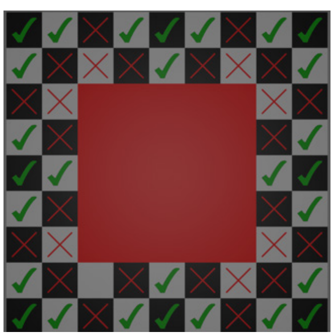


Figure 39: A 9 × 7 * 2 and 9 × 5 * 2 board placed next to each other with each circuit around the board coloured in a different colour to showcase how adding rows doesn't affect the patterns of the circuits.

Gallery of Boards:

Since the boards below show solutions for $(4k+1) \times (4l+1)$ and $(4k+1)\times(4l+3)$ boards, it can be simplified to say that a solution exists for every $(4k+1)\times(2l+1)$ board.

Figure 41 (shown below) shows solutions for both $4k \times (4l + 1) * 2$ boards and for $4k \times (4l + 3) * 2$ boards; this can be simplified to state that a solution exists for every $4k \times (2l + 1) * 2$ board



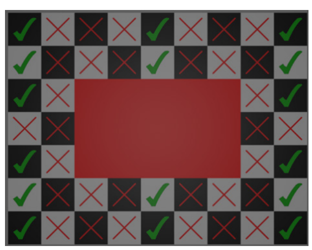
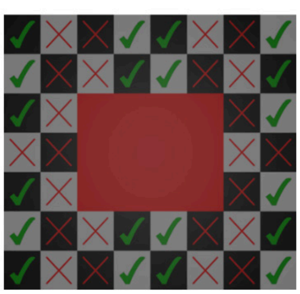


Figure 40: 9 × 9 board showing a solution for $(4k + 1) \times (4l + 1) * 2$ boards and 9 × 7 board showing solutions for $(4k + 1) \times (4l + 3) * 2$ boards



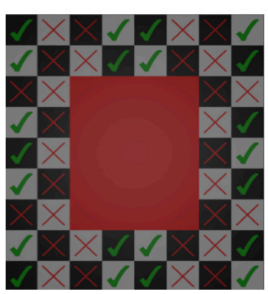


Figure 41: 8 × 7 * 2 showing a solution for $4k \times (4l + 3) * 2$ boards and 8 × 9 * 2 board showing a solution for $4k \times (4l + 1) * 2$ boards

General Hamiltonian Path Solutions for u × v*a Board Where a ≥ 3:

Theorem 2. A solution exists for $u \times v*a$ boards where $(v-3)/2 \geq 3 \geq a$ and $v \geq u$, however, other boards will have solutions:

Any board where the subdivided rectangles all have a Hamiltonian path will contain a Hamiltonian path. Essentially, the larger the board grows, the more likely a Hamiltonian board exists. As stated before, a $u \times v*a$ board will contain four rectangles of two sizes: two rectangles of size $u \times a$ and two rectangles of size $(v\text{-}2a) \times a$. As larger rectangles are much more likely to have a solution, $v \geq u \geq 7$, v must be greater than u so that the rectangle $v\text{-}2a \times a$ is as large as possible. The reason why both u and v must be ≥7 is because for $(u\text{-}2a) \times (v\text{-}2a)$ boards where $a \geq 3$, both $(u\text{-}2a)$ and $(v\text{-}2a)$ must be greater than 0, $\therefore$ $u,v \geq 7$. To prove a solution exists, it must be proven that $(v\text{-}2a) \times a$ and $u \times a$ boards have a solution. It is important to note that the following proof does not state every possible Hamiltonian path but states cases for which there will be Hamiltonian paths. There may be some boards that this proof fails to evaluate that other methods succeed in finding a Hamiltonian path for. As there are only solutions for m×n boards where $m \geq n \geq 3$, it can be generally stated that a solution exists when:

$v\text{-}2a \geq 3$

$v \geq 3+2a$

$(v\text{-}3)/2 \geq a$

Limitations of this solution:

The problem with this solution is that it is incomplete: some Hamiltonian Paths are not found or discovered that other methods of inference find. According to this formula, as $a \geq 3$ and a solution does not exist for a 3 × 3 board, $v \neq 9$ when a=3, it can be seen that v>9. For this method to be used when $a \geq 4, v \geq 11$, meaning the only time v=10, is for when a=3. Additionally, no solutions can be found when a=3 and v=11,12 as the solutions for the boards 3 × 5 and 3 × 6 do not exist. Finally, a solution doesn't exist for a=4 and v=12 as there are no solutions for 4 × 4 boards. Since a solution exists for every u×a board where $u \geq 7$ (and $a \geq 3$) there are no special cases to be concerned about. Thus, a Hamiltonian path exists on every $u \times v*a$ board where: $u \geq 7, a \geq 3$ and $v \geq 3+2a$, besides the boards $u \times (9,11,12) * 3$ and $u \times 12*4$.

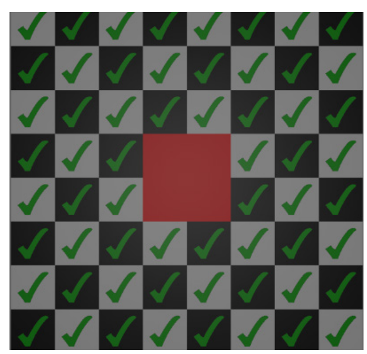


Figure 42: Solutions / Hamiltonian Paths for 8 × 8 * 2 board shown using CAP solver

There is an issue with the above solution, which is the fact that it's not categorically true for every single board. The above solution states that the boards 7 × 7*3, 7 × 8*3, 8 × 8*3, 9 × 8*3, 9 × 9*3 do not have a solution, contradicting the results produced by CAP for boards. To see this, the example board 8 × 8*3 was used. For this board, both u,v=3 and a=3, contradicting the statement that $v\text{-}2a \geq 3$ as 8-2(3)=2.

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Solutions for 3D Boards:

Theorem 7. A Hamiltonian path exists for every board of size $a \times b \times c$ where the boards $a \times b$,$a \times c$ and $b \times c$ exist:

This section will focus on 3D boards that can be represented as unfolded 2D boards. This will specifically generalize solutions for rectangular prisms but will give some standard ideas for other board shapes. Prism can be created by folding together a simple 2D shape template on a piece of paper (see Figures 44 and 45).

Figure 43: Unfolded 3D shapes[10]

Through this, a path around a 3D prism can be represented as multiple rectangles that can be linked by Hamiltonian paths. Special prisms such as triangular prisms can be proved to exist if a Hamiltonian path exists for whatever special face is being used. Note that this section will only consider the surfaces of the 3D objects. Specifically for this section, we will find the general cases in which a rectangular prism of size $a \times b \times c$ exists. A rectangular prism will always consist of three types of rectangles, each appearing twice. These rectangles will be of size $a \times b$,$a \times c$ and $b \times c$. Therefore, a solution will exist for a Hamiltonian path if a solution exists for the smaller rectangular boards.

To be clear for this section, this method will work when using a specific 2D shape that becomes a rectangular board due to the way connecting boards work. The problem with the Figure above is that some of the smaller rectangles would have to be traveled through twice for a solution to exist. Using the specific board shape shown below, smaller rectangles can have their Hamiltonian paths connected, proving a Hamiltonian path exists around every $a \times b \times c$ board where the boards of $a \times b$,$a \times c$ and $b \times c$ exist.

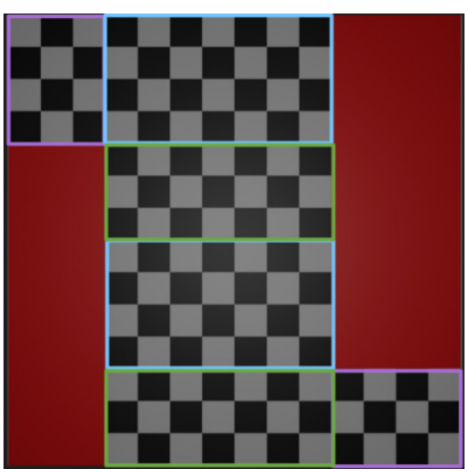
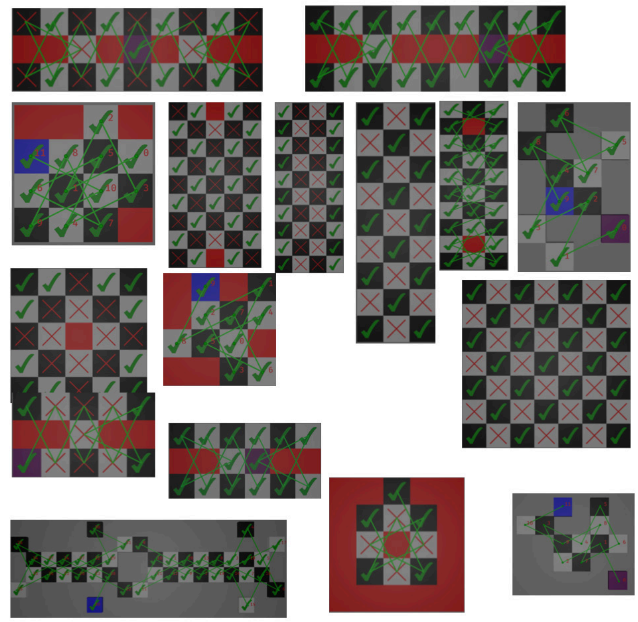


Figure 44: Unfolded rectangular prism that will create a cube of size 3 × 4 × 7.

Miscellaneous Board Observations Using CAP:



Conclusion

In this comprehensive study on Knight's Tours across a spectrum of chessboard configurations, we determined that for all n×n boards where n ≥ 4, and m × n boards with m, n ≥ 5, a path exists. Extending beyond conventional board layouts, the investigation delves into configurations with central rectangles removed, yielding solutions to these board arrangements and expanding the conventional understanding of Knight's Tour layouts. This study also covers several general theorems used to solve different board types. These theorems—ranging from specific solutions tailored for 3 × m, 4 × m, 5 × m, and 6 × m boards and their application to m×n boards later to complex 3D board theorems generated using 2D theorem foundations—provide a robust framework for understanding Hamiltonian paths in varied contexts. Collectively, these findings not only cement foundational principles in graph theory and combinatorics but also pave the way for novel methodologies and applications in these fields. This exploration stands as a testament to the intricate interplay between mathematical theory and practical application.

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