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RESEARCH ARTICLE

Converting a Radical Form $\sqrt[n]{a+\sqrt[n]{b}}$ into the Expression $\sqrt[n]{x}+\sqrt[n]{y}$ when n = 3, 4, 5, 6, and 7

Kyurin Kim[1], Yoonjae Lee [2], Sihu Park [3]

1. United World College of South East Asia Dover Campus, 1207 Dover Rd, Singapore, Singapore, 139654, Singapore; kim.lynn1202@gmail.com
2. Osaka International School of Kwansei Gakuin, 4-4-16 Onohara-nishi, Minoh-Shi, Osaka-fu, 562-0032, Japan; theyoonjaelee@gmail.com
3. Korea International School Jeju campus, 8 Global Edu-ro 260beon-gil, Daejeong-eup Seogwipo-si, Jejudo, 63644, Republic of Korea; coolpshu@gmail.com

ABSTRACT: This research delves into various ways of converting a radical of the form $\sqrt[n]{a+\sqrt[n]{b}}$ into the sum of two radicals $\sqrt[n]{x}+\sqrt[n]{y}$. The contents derive from the well-established conversion of rearranging the nested radical $\sqrt{a+\sqrt{b}}$ into the form $\sqrt{\frac{(a+\sqrt{a^2-b})}{2}}+\sqrt{\frac{(a-\sqrt{a^2-b})}{2}}$, which works as a denesting process being that $\sqrt{a^2-b}$ is a rational number. The motivation for this research was the intrinsic curiosity of wondering whether it would be possible to apply this expression for higher levels of power. We split the nested radical $\sqrt[n]{a+\sqrt[n]{b}}$ into the sum of two radicals $\sqrt[n]{x}+\sqrt[n]{y}$ by solving for x and y in terms of a and b for n values of 3, 4, 5, 6, and 7. However, as the values of n greater than 7 require the use of polynomial formulae greater than the cubic; we could not proceed further as it became too complex.

KEYWORDS: Mathematics, Algebra, Radical, Nested Form.

Introduction

Denesting a radical refers to the simplification of nested expressions for further mathematical calculations or analyses. One of the denesting conversion is rearranging the nested radical $\sqrt{a+\sqrt{b}}$ into the form $\sqrt{\frac{(a+\sqrt{a^2-b})}{2}}+\sqrt{\frac{(a-\sqrt{a^2-b})}{2}}$ when a^2-b is a square number. Inspired by this, we have studied ways to split the nested radical $\sqrt[n]{a+\sqrt[n]{b}}$ into the sum of two radicals $\sqrt[n]{x}+\sqrt[n]{y}$ by solving for x and y in terms of a and b, whether the $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are denested radicals or not; the radicality is determined by whether both x and y are simple integers, which yield denesting equations. If both x and y are not simple integers, they would not yield denesting equations. As we worked the transformations by increasing the values of n, it became more and more challenging, intriguing us further. For the values of n = 3, 4, and 5, the quadratic formula was used to solve for x and y in terms of positive numbers a and b. When the values of n = 6 and 7, we utilized not only the quadratic formula, but also the cubic formula with the help of the binomial theorem. However, we were not able to proceed further when $n>7$ because it required the use of polynomial formulae greater than that of the cubic, such as quartic or quintic formulae, which is simply too complicated.

Methods

These are some applications of how our formula works to each corresponding value of n.

• When n is 3, the equation is:

$$\sqrt[3]{a+\sqrt[3]{b}}=\sqrt[3]{\frac{a+\sqrt{a^2-\frac{4b}{27(a+\sqrt[3]{b})}}}{2}}+\sqrt[3]{\frac{a-\sqrt{a^2-\frac{4b}{27(a+\sqrt[3]{b})}}}{2}}$$

For example, if a and b are 25 and 8, respectively, which is 3:

$$3=\sqrt[3]{27}=\sqrt[3]{25+\sqrt[3]{8}},$$

$$=\sqrt[3]{\frac{25+\sqrt{625-\frac{32}{27(25+\sqrt[3]{8})}}}{2}}+\sqrt[3]{\frac{25-\sqrt{625-\frac{32}{27(25+\sqrt[3]{8})}}}{2}}$$

$$=\sqrt[3]{\frac{25+\sqrt{625-\frac{32}{27^2}}}{2}}+\sqrt[3]{\frac{25-\sqrt{625-\frac{32}{27^2}}}{2}}$$

$$=\sqrt[3]{\frac{25+\frac{\sqrt{455593}}{27}}{2}}+\sqrt[3]{\frac{25-\frac{\sqrt{455593}}{27}}{2}}$$

$$=\sqrt[3]{\frac{675+79\sqrt{73}}{54}}+\sqrt[3]{\frac{675-79\sqrt{73}}{54}}$$

• When n is 4, the equation is:

$$\sqrt[4]{a+\sqrt[4]{b}}=\sqrt[4]{\frac{a+\sqrt{a^2-4\left(\sqrt{a+\sqrt[4]{b}}-\sqrt{a+\frac{1}{2}\sqrt[4]{b}}\right)^4}}{2}}+\sqrt[4]{\frac{a-\sqrt{a^2-4\left(\sqrt{a+\sqrt[4]{b}}-\sqrt{a+\frac{1}{2}\sqrt[4]{b}}\right)^4}}{2}}$$

For example, if a and b are 78 and 81, respectively, which is also 3:

$$3=\sqrt[4]{81}=\sqrt[4]{78+\sqrt[4]{81}},$$

$$=\sqrt[4]{\frac{78+\sqrt{78^2-4\left(9-\sqrt{78+\frac{1}{2}\sqrt[4]{81}}\right)^4}}{2}}+\sqrt[4]{\frac{78-\sqrt{78^2-4\left(9-\sqrt{78+\frac{1}{2}\sqrt[4]{81}}\right)^4}}{2}}$$

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$$= \sqrt[4]{\frac{78 + \sqrt{6084 - 4\left(9 - \sqrt{78 + \frac{3}{2}}\right)^4}}{2}} + \sqrt[4]{\frac{78 - \sqrt{6084 - 4\left(9 - \sqrt{78 + \frac{3}{2}}\right)^4}}{2}}$$

$$= \sqrt[4]{\frac{78 + \sqrt{6084 - 4\left(9 - \sqrt{\frac{159}{2}}\right)^4}}{2}} + \sqrt[4]{\frac{78 - \sqrt{6084 - 4\left(9 - \sqrt{\frac{159}{2}}\right)^4}}{2}}$$

- When n is 5, the formula is:

$$\sqrt[5]{a + \sqrt[5]{b}} = \sqrt[5]{\frac{(a + \sqrt{a^2 - 4\left(\frac{5\sqrt[5]{(a + \sqrt[5]{b})^3} - \sqrt{5\sqrt[5]{a + \sqrt[5]{b}}(5a + \sqrt[5]{b})}}{10\sqrt[5]{a + \sqrt[5]{b}}}\right)^5}}{2}} + \sqrt[5]{\frac{(a - \sqrt{a^2 - 4\left(\frac{5\sqrt[5]{(a + \sqrt[5]{b})^3} - \sqrt{5\sqrt[5]{a + \sqrt[5]{b}}(5a + \sqrt[5]{b})}}{10\sqrt[5]{a + \sqrt[5]{b}}}\right)^5}}{2}}$$

For example, if a and b are 241 and 32, respectively, which is 3 as well:

$$3 = \sqrt[5]{243} = \sqrt[5]{241 + \sqrt[5]{32}},$$

$$= \sqrt[5]{\frac{241 + \sqrt{241^2 - 4\left(\frac{5\sqrt[5]{(241 + \sqrt[5]{32})^3} - \sqrt{5\sqrt[5]{241 + \sqrt[5]{32}}(5(241) + \sqrt[5]{32})}}{10\sqrt[5]{241 + \sqrt[5]{32}}}\right)^5}}{2}} + \sqrt[5]{\frac{241 - \sqrt{241^2 - 4\left(\frac{5\sqrt[5]{(241 + \sqrt[5]{32})^3} - \sqrt{5\sqrt[5]{241 + \sqrt[5]{32}}(5(241) + \sqrt[5]{32})}}{10\sqrt[5]{241 + \sqrt[5]{32}}}\right)^5}}{2}}$$

$$= \sqrt[5]{\frac{241 + \sqrt{58081 - 4\left(\frac{135 - \sqrt{15(1205 + 2)}}{30}\right)^5}}{2}} + \sqrt[5]{\frac{241 - \sqrt{58081 - 4\left(\frac{135 - \sqrt{15(1205 + 2)}}{30}\right)^5}}{2}}$$

$$= \sqrt[5]{\frac{241 + \sqrt{58081 - 4\left(\frac{135 - \sqrt{18105}}{30}\right)^5}}{2}} + \sqrt[5]{\frac{241 - \sqrt{58081 - 4\left(\frac{135 - \sqrt{18105}}{30}\right)^5}}{2}}$$

- When n is 6, the formula is:

$$\sqrt[6]{a + \sqrt[6]{b}} = \sqrt[6]{\frac{a + \sqrt{a^2 - 4k^6}}{2}} + \sqrt[6]{\frac{a - \sqrt{a^2 - 4k^6}}{2}}$$

where k:

$$\frac{3}{2}\sqrt[3]{a + \sqrt[6]{b}} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a - 594\sqrt[6]{b} + \sqrt{(-486a - 594\sqrt[6]{b})^2 - 364500(a + \sqrt[6]{b})^2}\right)} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a - 594\sqrt[6]{b} - \sqrt{(-486a - 594\sqrt[6]{b})^2 - 364500(a + \sqrt[6]{b})^2}\right)}$$

$$\sqrt[6]{2 + \sqrt[6]{3}}$$

For example, if a and b are 2 and 3, respectively, then the k value is:

$$\frac{3}{2}\sqrt[3]{2 + \sqrt[6]{3}} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-972 - 594\sqrt[6]{3} + \sqrt{(-972 - 594\sqrt[6]{3})^2 - 364500(2 + \sqrt[6]{3})^2}\right)} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-972 - 594\sqrt[6]{3} - \sqrt{(-972 - 594\sqrt[6]{3})^2 - 364500(2 + \sqrt[6]{3})^2}\right)}$$

$$= \sqrt[6]{\frac{2 + \sqrt{2^2 - 4k^6}}{2}} + \sqrt[6]{\frac{2 - \sqrt{2^2 - 4k^6}}{2}}$$

$$= \sqrt[6]{1 + \sqrt{1 - k^6}} + \sqrt[6]{1 - \sqrt{1 - k^6}},$$

Therefore, the final expression is

$$\sqrt[6]{1 + \sqrt{1 - \left(\frac{3}{2}\sqrt[3]{2 + \sqrt[6]{3}} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-972 - 594\sqrt[6]{3} + \sqrt{(-972 - 594\sqrt[6]{3})^2 - 364500(2 + \sqrt[6]{3})^2}\right)} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-972 - 594\sqrt[6]{3} - \sqrt{(-972 - 594\sqrt[6]{3})^2 - 364500(2 + \sqrt[6]{3})^2}\right)}\right)^6}}$$
$$+ \sqrt[6]{1 - \sqrt{1 - \left(\frac{3}{2}\sqrt[3]{2 + \sqrt[6]{3}} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-972 - 594\sqrt[6]{3} + \sqrt{(-972 - 594\sqrt[6]{3})^2 - 364500(2 + \sqrt[6]{3})^2}\right)} - \frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-972 - 594\sqrt[6]{3} - \sqrt{(-972 - 594\sqrt[6]{3})^2 - 364500(2 + \sqrt[6]{3})^2}\right)}\right)^6}}$$

- When n is 7, the formula is:

$$\sqrt[7]{a + \sqrt[7]{b}} = \sqrt[7]{\frac{a + \sqrt{a^2 - 4k^7}}{2}} + \sqrt[7]{\frac{a - \sqrt{a^2 - 4k^7}}{2}}$$

Where $k = \frac{2}{3}\sqrt[7]{(a + \sqrt[7]{b})^2}$

$$-\frac{1}{21\sqrt[7]{a + \sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(a + \sqrt[7]{b})^9} - 1323\sqrt[7]{(a + \sqrt[7]{b})^2 b} + \sqrt{\left(686\sqrt[7]{(a + \sqrt[7]{b})^9} - 1323\sqrt[7]{(a + \sqrt[7]{b})^2 b}\right)^2 - 4\left(196\sqrt[7]{(a + \sqrt[7]{b})^6} + 21\sqrt[7]{(a + \sqrt[7]{b})b}\right)^3}\right)}$$

$$-\frac{1}{21\sqrt[7]{a + \sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(a + \sqrt[7]{b})^9} - 1323\sqrt[7]{(a + \sqrt[7]{b})^2 b} - \sqrt{\left(686\sqrt[7]{(a + \sqrt[7]{b})^9} - 1323\sqrt[7]{(a + \sqrt[7]{b})^2 b}\right)^2 - 4\left(196\sqrt[7]{(a + \sqrt[7]{b})^6} + 21\sqrt[7]{(a + \sqrt[7]{b})b}\right)^3}\right)}$$

$$\sqrt[7]{2 + \sqrt[7]{3}}$$

For example, if a and b are 2 and 3, respectively, then the k value is:

$$\frac{2}{3}\sqrt[7]{(2 + \sqrt[7]{3})^2}$$

$$-\frac{1}{21\sqrt[7]{2 + \sqrt[7]{3}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3} + \sqrt{\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3}\right)^2 - 4\left(196\sqrt[7]{(2 + \sqrt[7]{3})^6} + 21\sqrt[7]{(2 + \sqrt[7]{3})3}\right)^3}\right)}$$

$$-\frac{1}{21\sqrt[7]{2 + \sqrt[7]{3}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3} - \sqrt{\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3}\right)^2 - 4\left(196\sqrt[7]{(2 + \sqrt[7]{3})^6} + 21\sqrt[7]{(2 + \sqrt[7]{3})3}\right)^3}\right)}$$

$$= \sqrt[7]{\frac{2 + \sqrt{4 - 4k^7}}{2}} + \sqrt[7]{\frac{2 - \sqrt{4 - 4k^7}}{2}}$$

$$= \sqrt[7]{1 + \sqrt{1 - k^7}} + \sqrt[7]{1 - \sqrt{1 - k^7}},$$

Therefore, the final expression is:

$$= \sqrt[7]{1 + \sqrt{1 - \left(\frac{2}{3}\sqrt[7]{(2 + \sqrt[7]{3})^2} - \frac{1}{21\sqrt[7]{2 + \sqrt[7]{3}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3} + \sqrt{\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3}\right)^2 - 4\left(196\sqrt[7]{(2 + \sqrt[7]{3})^6} + 21\sqrt[7]{(2 + \sqrt[7]{3})3}\right)^3}\right)} - \frac{1}{21\sqrt[7]{2 + \sqrt[7]{3}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3} - \sqrt{\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3}\right)^2 - 4\left(196\sqrt[7]{(2 + \sqrt[7]{3})^6} + 21\sqrt[7]{(2 + \sqrt[7]{3})3}\right)^3}\right)}\right)^7}}$$
$$+ \sqrt[7]{1 - \sqrt{1 - \left(\frac{2}{3}\sqrt[7]{(2 + \sqrt[7]{3})^2} - \frac{1}{21\sqrt[7]{2 + \sqrt[7]{3}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3} + \sqrt{\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3}\right)^2 - 4\left(196\sqrt[7]{(2 + \sqrt[7]{3})^6} + 21\sqrt[7]{(2 + \sqrt[7]{3})3}\right)^3}\right)} - \frac{1}{21\sqrt[7]{2 + \sqrt[7]{3}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3} - \sqrt{\left(686\sqrt[7]{(2 + \sqrt[7]{3})^9} - 1323\sqrt[7]{(2 + \sqrt[7]{3})^2 3}\right)^2 - 4\left(196\sqrt[7]{(2 + \sqrt[7]{3})^6} + 21\sqrt[7]{(2 + \sqrt[7]{3})3}\right)^3}\right)}\right)^7}}$$

Result and Discussion

The following are the steps to split the nested radical $\sqrt[n]{a + \sqrt[n]{b}}$ into the sum of two radicals such as $\sqrt[n]{x} + \sqrt[n]{y}$, where x and y would be expressed in terms of positive numbers a and b.

Case 1. When n=3

First, set up an equation:

$$\sqrt[3]{a + \sqrt[3]{b}} = \sqrt[3]{x} + \sqrt[3]{y} \tag{1}$$

Cube the equation (1)

$$a + \sqrt[3]{b} = x + 3\sqrt[3]{x^2 y} + 3\sqrt[3]{xy^2} + y$$

While x+y is always rational, $\sqrt[3]{b}$ can occur as both rational and irrational due to certain values of b where the decomposed values of nth root b become rational.

$$\begin{cases} a = x + y \\ \sqrt[3]{b} = 3\sqrt[3]{x^2 y} + 3\sqrt[3]{xy^2} \end{cases} \tag{2}$$

The second equation of (2) can be factored.

$$\sqrt[3]{b} = 3\sqrt[3]{xy}\left(\sqrt[3]{x} + \sqrt[3]{y}\right)$$

By the equation (1), the above equation can be organized as

$$\sqrt[3]{b} = 3\sqrt[3]{xy} \cdot \sqrt[3]{a + \sqrt[3]{b}}$$

Then, xy is expressed as

$$xy = \frac{b}{27(a + \sqrt[3]{b})} \tag{3}$$

Using the equations (2) and (3), x and y can be solved for in terms of a and b using the quadratic formula.

$$y = a - x \tag{4}$$

Plug in the above equation to (3)

$$x(a - x) = \frac{b}{27(a + \sqrt[3]{b})}$$

$$\Rightarrow x^2 - ax + \frac{b}{27(a + \sqrt[3]{b})} = 0$$

$$x = \frac{a \pm \sqrt{a^2 - \frac{4b}{27(a + \sqrt[3]{b})}}}{2}$$

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By plugging the equation above into (4),

$$y = a - \frac{a \pm \sqrt{a^2 - \frac{4b}{27(a+\sqrt[3]{b})}}}{2}$$

$$y = \frac{a \mp \sqrt{a^2 - \frac{4b}{27(a+\sqrt[3]{b})}}}{2}$$

The order of plus and minus in each expression of x and y does not make a difference if they have alternate signs since they are added together.

Therefore,

$$\sqrt[3]{\boldsymbol{a}+\sqrt[3]{\boldsymbol{b}}} = \sqrt[3]{\frac{\boldsymbol{a}+\sqrt{\boldsymbol{a}^2 - \frac{4\boldsymbol{b}}{27(\boldsymbol{a}+\sqrt[3]{\boldsymbol{b}})}}}{2}} + \sqrt[3]{\frac{\boldsymbol{a}-\sqrt{\boldsymbol{a}^2 - \frac{4\boldsymbol{b}}{27(\boldsymbol{a}+\sqrt[3]{\boldsymbol{b}})}}}{2}}$$

Case 2. When n=4

Start with setting up an equation:

$$\sqrt[4]{a+\sqrt[4]{b}} = \sqrt[4]{x} + \sqrt[4]{y} \tag{5}$$

Power of 4 the equation (5)

$$a + \sqrt[4]{b} = x + 4\sqrt[4]{x^3y} + 6\sqrt[4]{x^2y^2} + 4\sqrt[4]{xy^3} + y$$

While $x+y$ is always rational, $\sqrt[3]{b}$ can occur as both rational and irrational due to certain values of b where the decomposed values of nth root b become rational.

$$\begin{cases} a = x + y \\ \sqrt[4]{b} = 4\sqrt[4]{x^3y} + 6\sqrt[4]{x^2y^2} + 4\sqrt[4]{xy^3} \end{cases} \tag{6}$$

The second equation of (6) can be factored.

$$\sqrt[4]{b} = 2\sqrt[4]{xy}\left(2\sqrt[4]{x^2} + 3\sqrt[4]{xy} + 2\sqrt[4]{y^2}\right)$$

As $2\sqrt[4]{x^2} + 3\sqrt[4]{xy} + 2\sqrt[4]{y^2}$ is identical to $2\left(\sqrt[4]{x} + \sqrt[4]{y}\right)^2 - \sqrt[4]{xy}$, the equation above can be expressed as:

$$\sqrt[4]{b} = 2\sqrt[4]{xy}\left(2\left(\sqrt[4]{x} + \sqrt[4]{y}\right)^2 - \sqrt[4]{xy}\right)$$

By the equation (5),

$$\sqrt[4]{b} = 2\sqrt[4]{xy}\left(2\sqrt[4]{\left(a+\sqrt[4]{b}\right)}^2 - \sqrt[4]{xy}\right)$$

$$\rightarrow 0 = 2\sqrt[4]{xy}^2 - 4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \cdot \sqrt[4]{xy} + \sqrt[4]{b}$$

Apply the quadratic formula to solve for $\sqrt[4]{xy}$

$$\sqrt[4]{xy} = \frac{2\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \pm \sqrt{4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^4} - 2\sqrt[4]{b}}}{2}$$

$$\rightarrow xy = \left(\frac{2\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \pm \sqrt{4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^4} - 2\sqrt[4]{b}}}{2}\right)^4 \tag{7}$$

Using (6) and (7), x and y are expressed:

$$x(a-x) = \left(\frac{2\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \pm \sqrt{4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^4} - 2\sqrt[4]{b}}}{2}\right)^4$$

$$\rightarrow 0 = x^2 - ax + \left(\frac{2\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \pm \sqrt{4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^4} - 2\sqrt[4]{b}}}{2}\right)^4$$

Therefore,

$$x = \frac{a + \sqrt{a^2 - 4\left(\frac{2\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \pm \sqrt{4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^4} - 2\sqrt[4]{b}}}{2}\right)^4}}{2}$$

$$y = \frac{a - \sqrt{a^2 - 4\left(\frac{2\sqrt[4]{\left(a+\sqrt[4]{b}\right)^2} \pm \sqrt{4\sqrt[4]{\left(a+\sqrt[4]{b}\right)^4} - 2\sqrt[4]{b}}}{2}\right)^4}}{2}$$

Finally, x and y are substituted into the initial equation (5).

$$\sqrt[4]{a+\sqrt[4]{b}} = \sqrt[4]{\frac{a + \sqrt{a^2 - 4\left(\frac{2\sqrt{a+\sqrt[4]{b}} \pm \sqrt{4(a+\sqrt[4]{b}) - 2\sqrt[4]{b}}}{2}\right)^4}}{2}} + \sqrt[4]{\frac{a - \sqrt{a^2 - 4\left(\frac{2\sqrt{a+\sqrt[4]{b}} \pm \sqrt{4(a+\sqrt[4]{b}) - 2\sqrt[4]{b}}}{2}\right)^4}}{2}}$$

$$\rightarrow \sqrt[4]{a+\sqrt[4]{b}} = \sqrt[4]{\frac{a + \sqrt{a^2 - 4\left(\sqrt{a+\sqrt[4]{b}} \pm \sqrt{a+\frac{1}{2}\sqrt[4]{b}}\right)^4}}{2}} + \sqrt[4]{\frac{a - \sqrt{a^2 - 4\left(\sqrt{a+\sqrt[4]{b}} \pm \sqrt{a+\frac{1}{2}\sqrt[4]{b}}\right)^4}}{2}}$$

As (a,b) is a positive integer pair, $\sqrt[4]{a+\sqrt[4]{b}}$ cannot be complex. Cases where $4\left(\sqrt{a+\sqrt[4]{b}} \pm \sqrt{a+\frac{1}{2}\sqrt[4]{b}}\right)^4$ is larger than a^2 would produce complex equations, and should be not be counted.

Case 3. When n=5

Start by setting up an equation:

$$\sqrt[5]{a+\sqrt[5]{b}} = \sqrt[5]{x} + \sqrt[5]{y} \tag{8}$$

5th power the equation (8)

$$a + \sqrt[5]{b} = x + y + 5\sqrt[5]{x^4y} + 10\sqrt[5]{x^3y^2} + 10\sqrt[5]{x^2y^3} + 5\sqrt[5]{xy^4}$$

While x+y is always rational, $\sqrt[3]{b}$ can occur as both rational and irrational due to certain values of b where the decomposed values of nth root b become rational.

$$\begin{cases} a = x + y \\ \sqrt[5]{b} = 5\sqrt[5]{x^4y} + 10\sqrt[5]{x^3y^2} + 10\sqrt[5]{x^2y^3} + 5\sqrt[5]{xy^4} \end{cases} \tag{9}$$

The second equation of (9) can be factored:

$$\sqrt[5]{b} = 5\sqrt[5]{xy} \cdot \left(\sqrt[5]{x^3} + 2 \cdot \sqrt[5]{x^2y} + 2\sqrt[5]{xy^2} + \sqrt[5]{y^3}\right)$$

As $(\sqrt[5]{x} + \sqrt[5]{y})^3$ is equivalent to $\sqrt[5]{x^3} + 3\sqrt[5]{x^2y} + 3\sqrt[5]{xy^2} + \sqrt[5]{y^3}$ the equation can be expressed as

$$\sqrt[5]{b} = 5\sqrt[5]{xy} \cdot \left(\sqrt[5]{(a+\sqrt[5]{b})^3} - \left(\sqrt[5]{x^2y} + \sqrt[5]{xy^2}\right)\right)$$

$$= 5\sqrt[5]{xy} \cdot \left(\sqrt[5]{(a+\sqrt[5]{b})^3} - \sqrt[5]{xy}\left(\sqrt[5]{x} + \sqrt[5]{y}\right)\right)$$

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By the original equation (8),

$$= 5\sqrt[5]{xy}\cdot\left(\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}-\sqrt[5]{xy}\left(\sqrt[5]{a+\sqrt[5]{b}}\right)\right)$$

$$\rightarrow 5\left(\sqrt[5]{a+\sqrt[5]{b}}\right)\left(\sqrt[5]{xy}\right)^2-5\left(\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\right)\left(\sqrt[5]{xy}\right)+\sqrt[5]{b}=0$$

Apply the quadratic formula to solve for $\sqrt[5]{xy}$

$$\sqrt[5]{xy}=\frac{5\sqrt[5]{a+\sqrt[5]{b}}^3\pm\sqrt{25\sqrt[5]{a+\sqrt[5]{b}}^6-4\left(5\sqrt[5]{a+\sqrt[5]{b}}\right)\left(\sqrt[5]{b}\right)}}{10\sqrt[5]{a+\sqrt[5]{b}}}$$

$$\sqrt[5]{xy}=\frac{5\cdot\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\cdot\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\sqrt[5]{a+\sqrt[5]{b}}}$$

$$\rightarrow xy=\left(\frac{5\cdot\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\cdot\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\cdot\sqrt[5]{a+\sqrt[5]{b}}}\right)^5 \tag{10}$$

Using (9) and (10), x and y are expressed

$$x(a-x)=\left(\frac{5\cdot\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\cdot\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\cdot\sqrt[5]{a+\sqrt[5]{b}}}\right)^5$$

$$x^2-ax+\left(\frac{5\cdot\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\cdot\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\cdot\sqrt[5]{a+\sqrt[5]{b}}}\right)^5=0$$

Therefore,

$$x=\frac{a+\sqrt{a^2-4\left(\frac{5\cdot\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\cdot\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\cdot\sqrt[5]{a+\sqrt[5]{b}}}\right)^5}}{2}$$

$$y=\frac{a-\sqrt{a^2-4\left(\frac{5\cdot\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\cdot\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\cdot\sqrt[5]{a+\sqrt[5]{b}}}\right)^5}}{2}$$

Finally, x and y are substituted into the initial equation (8).

$$\sqrt[5]{a+\sqrt[5]{b}}=\sqrt[5]{\frac{\left(a+\sqrt{a^2-4\left(\frac{5\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\sqrt[5]{a+\sqrt[5]{b}}}\right)^5}\right.}{2}}+\sqrt[5]{\frac{\left(a-\sqrt{a^2-4\left(\frac{5\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\sqrt[5]{a+\sqrt[5]{b}}}\right)^5}\right.}{2}}$$

As (a,b) is a positive integer pair, $\sqrt[5]{a+\sqrt[5]{b}}$ cannot be complex. Cases where $4\left(\frac{5\sqrt[5]{\left(a+\sqrt[5]{b}\right)^3}\pm\sqrt{5\sqrt[5]{a+\sqrt[5]{b}}\left(5a+\sqrt[5]{b}\right)}}{10\sqrt[5]{a+\sqrt[5]{b}}}\right)^5$ is larger than a^2 would produce complex equations, and should be not be counted.

Case 4. When n=6

Start by setting up an equation:

$$\sqrt[6]{a+\sqrt[6]{b}}=\sqrt[6]{x}+\sqrt[6]{y} \tag{11}$$

6th power the equation (11)

$$a+\sqrt[6]{b}=x+y+6\sqrt[6]{x^5y}+15\sqrt[6]{x^4y^2}+20\sqrt[6]{x^3y^3}+15\sqrt[6]{x^2y^4}+6\sqrt[6]{xy^5}$$

While x+y is always rational, $\sqrt[3]{}$b can occur as both rational and irrational due to certain values of b where the decomposed values of nth root b become rational.

$$\begin{cases} a=x+y \\ \sqrt[6]{b}=\sqrt[6]{xy}\left(6\sqrt[6]{x^4}+15\sqrt[6]{x^3y}+20\sqrt[6]{x^2y^2}+15\sqrt[6]{xy^3}+6\sqrt[6]{y^4}\right) \end{cases} \tag{12}$$

By using the powered outcomes of equation (11):

$$\begin{cases} 6\sqrt[6]{\left(a+\sqrt[6]{b}\right)^4}=6\sqrt[6]{x^4}+24\sqrt[6]{x^3y}+36\sqrt[6]{x^2y^2}+24\sqrt[6]{xy^3}+6\sqrt[6]{y^4} \\ 9\sqrt[6]{\left(a+\sqrt[6]{b}\right)^2}=9\sqrt[6]{x^2}+18\sqrt[6]{xy}+9\sqrt[6]{y^2} \\ \sqrt[6]{a+\sqrt[6]{b}}=\sqrt[6]{x}+\sqrt[6]{y} \end{cases}$$

The second equation of (12) can be organized with the steps below:

$$\sqrt[6]{b}=\sqrt[6]{xy}\left(6\sqrt[6]{\left(a+\sqrt[6]{b}\right)^4}-\left(9\sqrt[6]{x^3y}+16\sqrt[6]{x^2y^2}+9\sqrt[6]{xy^3}\right)\right)$$

$$\sqrt[6]{b}=\sqrt[6]{xy}\left(6\sqrt[6]{\left(a+\sqrt[6]{b}\right)^4}-\sqrt[6]{xy}\left(9\sqrt[6]{x^2}+16\sqrt[6]{xy}+9\sqrt[6]{y^2}\right)\right)$$

$$\sqrt[6]{b}=\sqrt[6]{xy}\left(6\sqrt[6]{\left(a+\sqrt[6]{b}\right)^4}-\sqrt[6]{xy}\left(9\sqrt[6]{\left(a+\sqrt[6]{b}\right)^2}-2\sqrt[6]{xy}\right)\right)$$

$$\rightarrow 2\left(\sqrt[6]{xy}\right)^3-9\sqrt[3]{a+\sqrt[6]{b}}\left(\sqrt[6]{xy}\right)^2+6\sqrt[3]{\left(a+\sqrt[6]{b}\right)^2}\left(\sqrt[6]{xy}\right)-\sqrt[6]{b}=0$$

Apply the cubic formula to solve for real number $\sqrt[6]{xy}$,[5]

$$\sqrt[6]{xy}=\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)} \tag{13}$$

Using (12) and (13), x and y can be expressed.

$$x(a-x)=\left(\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}\right)^6$$

$$\rightarrow x^2-ax+\left(\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}\right)^6=0$$

Therefore,

$$x=\frac{a+\sqrt{a^2-4\left(\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}\right)^6}}{2}$$

$$y=\frac{a-\sqrt{a^2-4\left(\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}\right)^6}}{2}$$

Finally, x and y are substituted into the initial equation (11).

$$\sqrt[6]{a+\sqrt[6]{b}}=$$

$$\sqrt[6]{\frac{a+\sqrt{a^2-4\left(\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}\right)^6}}{2}}$$

$$+\sqrt[6]{\frac{a-\sqrt{a^2-4\left(\frac{3}{2}\sqrt[3]{a+\sqrt[6]{b}}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}+\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}-\frac{1}{6}\sqrt[3]{\frac{1}{2}\left(-486a-594\sqrt[6]{b}-\sqrt{\left(-486a-594\sqrt[6]{b}\right)^2-364500\left(a+\sqrt[6]{b}\right)^2}\right)}\right)^6}}{2}}$$

Case 5. When n=7

Start by setting up an equation:

$$\sqrt[7]{a+\sqrt[7]{b}}=\sqrt[7]{x}+\sqrt[7]{y} \tag{14}$$

7th power the equation (14):

$$a+\sqrt[7]{b}=x+y+7\sqrt[7]{xy^6}+21\sqrt[7]{x^2y^5}+35\sqrt[7]{x^3y^4}+35\sqrt[7]{x^4y^3}+21\sqrt[7]{x^5y^2}+7\sqrt[7]{x^6y}$$

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While x+y is always rational, $\sqrt[3]{b}$ can occur as both rational and irrational due to certain values of b where the decomposed values of nth root b become rational.

$$\begin{cases} a = x + y \\ \sqrt[7]{b} = 7\sqrt[7]{xy}\left(\sqrt[7]{x^5} + 3\sqrt[7]{xy^4} + 5\sqrt[7]{x^2y^3} + 5\sqrt[7]{x^3y^2} + 3\sqrt[7]{x^4y} + \sqrt[7]{y^5}\right) \end{cases} \tag{15}$$

By using the powered outcomes of equation (14):

$$\begin{cases} \sqrt[7]{\left(a + \sqrt[7]{b}\right)^5} = \sqrt[7]{x^5} + 5\sqrt[7]{xy^4} + 10\sqrt[7]{x^2y^3} + 10\sqrt[7]{x^3y^2} + 5\sqrt[6]{x^4y} + \sqrt[7]{y^5} \\ \sqrt[7]{\left(a + \sqrt[7]{b}\right)^3} = \sqrt[7]{x^3} + 3\sqrt[7]{xy^2} + 3\sqrt[7]{x^2y} + \sqrt[7]{y^3} \\ \sqrt[7]{a + \sqrt[7]{b}} = \sqrt[7]{x} + \sqrt[7]{y} \end{cases}$$

The second equation of (15) can be organized with the steps below:

$$\sqrt[7]{b} = 7\sqrt[7]{xy}(\sqrt[7]{\left(a + \sqrt[7]{b}\right)^5} - \sqrt[7]{xy}\left(2\sqrt[7]{x^3} + 5\sqrt[7]{xy^2} + 5\sqrt[7]{x^2y} + 2\sqrt[7]{y^3}\right)$$

$$\sqrt[7]{b} = 7\sqrt[7]{xy}\left(\sqrt[7]{\left(a + \sqrt[7]{b}\right)^5} - \sqrt[7]{xy}\left(2\sqrt[7]{\left(a + \sqrt[7]{b}\right)^3} - \sqrt[7]{x^2y} - \sqrt[7]{xy^2}\right)\right)$$

$$\sqrt[7]{b} = 7\sqrt[7]{xy}\left(\sqrt[7]{\left(a + \sqrt[7]{b}\right)^5} - \sqrt[7]{xy}\left(2\sqrt[7]{\left(a + \sqrt[7]{b}\right)^3} - \sqrt[7]{xy}\sqrt[7]{a + \sqrt[7]{b}}\right)\right)$$

$$\rightarrow 7\sqrt[7]{a + \sqrt[7]{b}}\left(\sqrt[7]{xy}\right)^3 - 14\sqrt[7]{\left(a + \sqrt[7]{b}\right)^3}\left(\sqrt[7]{xy}\right)^2 + 7\sqrt[7]{\left(a + \sqrt[7]{b}\right)^5}\sqrt[7]{xy} - \sqrt[7]{b} = 0$$

Apply the cubic formula to solve for real number $\sqrt[7]{xy}$,[5]

$$\sqrt[7]{xy} = \frac{2}{3}\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b + \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)}$$
$$\frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b - \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)} \tag{16}$$

Using (15) and (16), x and y can be expressed.

$$x(a-x) = \left(\frac{2}{3}\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b + \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b - \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)}\right)^7$$

$$\rightarrow x^2 - ax + \left(\frac{2}{3}\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b + \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b - \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)}\right)^7 = 0$$

Therefore,

$$x = \frac{a + \sqrt{a^2 - 4\left(\frac{2}{3}\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b + \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b - \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)}\right)^7}}{2}$$

$$y = \frac{a - \sqrt{a^2 - 4\left(\frac{2}{3}\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b + \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)} - \frac{1}{21\sqrt[7]{a+\sqrt[7]{b}}}\sqrt[3]{\frac{1}{2}\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2}b - \sqrt{\left(686\sqrt[7]{\left(a+\sqrt[7]{b}\right)^9} - 1323\sqrt[7]{\left(a+\sqrt[7]{b}\right)^2b}\right)^2 - 4\left(196\sqrt[7]{\left(a+\sqrt[7]{b}\right)^6} + 21\sqrt[7]{\left(a+\sqrt[7]{b}\right)b}\right)^3}\right)}\right)^7}}{2}$$

Finally, x and y are substituted into the initial equation (15).

Conclusion

In this study, we showed ways to split the nested radical $\sqrt[n]{a + \sqrt[n]{b}}$ into the sum of two radicals $\sqrt[n]{x} + \sqrt[n]{y}$ when n=3, 4, 5, 6, and 7, whether $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are denested radicals or not. First, both sides of the equation are powered by n, which removes the big n-th root and expands the $\sqrt[n]{x} + \sqrt[n]{y}$ form. Then, the values of xy and x+y were worked out in terms of a and b by using substitution and the binomial theorem, which was especially helpful when finding the value of xy for n=6 and 7 because we needed to apply the cubic formula. This would allow us to develop a further enhanced understanding of algebraic radicals and serve as a formula for these specific cases. While a prevailing notion, this concept is not often covered in mathematical literature. We believe that this series of formulae can continue for infinity with n being any real number, and we wish to explore this further by experimenting with three or more variables such as $\sqrt[n]{x} + \sqrt[n]{y} + \sqrt[n]{z}$. While the expressions obtained typically introduce more complicated expressions, cases such as our examples exist where particular choices of a and b may simplify the original radical. A thorough investigation of these conditions, although beyond the scope of our current knowledge, remains a prospective topic for future research.

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Authors

Kyurin (Lynn) Kim, a passionate individual with a great interest in mathematics, is a freshman in UWCSEA Dover. As an elected student body representative and member of the biochemistry society, she is very active in the community and aspires to pursue a carrier in biology, particularly in the field of medicine.

Yoonjae Lee, an avid student of mathematics competitions with a keen interest in economics, is a sophomore at OIS. He collaborates with members of the Economics club to integrate mathematical principles with economics-related data, creating formulas and quantifying them. In the future, He wants to conduct innovative research in data analysis and finance.

Sihu Park, a devoted individual with passionate enthusiasm in the field of mathematics and physics, is a freshman at KIS. Having chosen the areas of interest to be mechanical engineering, he now delves deeper into studies relating to physics and applied mathematics.

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