

From Stars to Stairs: Multichoose, Generating Functions, and a Visual Bijection

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ABSTRACT: We present a combinatorial reinterpretation of the concept of combinations with repetition—also known as multicombinations—using non-decreasing *stair sequences* (i.e., sequences of steps with non-decreasing heights). This viewpoint provides an intuitive geometric picture analogous to the classical stars-and-bars approach and naturally leads to a *two-variable generating function* that generalizes the ordinary generating function for combinations. We derive an identity connecting this new generating function for multicombinations to the standard one for ordinary combinations (without repetition) and discuss connections to known results in combinatorics and physics. Historical context and illustrative examples are included to maintain a didactic yet rigorous exposition accessible to students.

KEYWORDS: Mathematics, Combinatorics, Combination with Repetition, Staircase Analogy, Generating Functions.

Introduction

Combinatorics is the art of counting the number of ways discrete objects can be arranged or selected under various rules. Whether we're assigning roles, forming teams, or choosing pizza toppings, understanding when the order matters or when repeated choices are allowed leads to different counting strategies: permutations account for ordered selections; *combinations* ignore order; and *multicombinations*, also known as combinations with repetition, allow repeated choices. These ideas underpin a wide range of problems in mathematics, science, and engineering.

For example, out of $N = 10$ students in a class, choosing a president, vice president, and secretary involves permutations: there are $P(10, 3) = \frac{10!}{(10-3)!} = 720$ possible ordered selections. By contrast, choosing any 3 students to form a committee (with no specific roles) involves combinations, such that the number of possible outcomes is $C(10, 3) = \binom{10}{3} = \frac{10!}{3!7!} = 120$. Furthermore, suppose a pizzeria provides $N = 10$ different pizza toppings, and customers can choose $L = 3$ toppings, allowing repetition (e.g., one could take a double or triple portion of a certain topping). This scenario corresponds to multicomination, i.e., selecting items with repetition allowed. The counting formula for this situation, derived via a classic stars-and-bars argument (to be explained below), is $\binom{L+N-1}{L} = \frac{(L+N-1)!}{L!(N-1)!}$. For $N = 10$ and $L = 3$ this yields $\binom{12}{3} = 220$ ways of choosing 3 toppings from 10 types with repetition allowed.

Table 1: All 10 multicombinations of 3 fruits (A = apple, B = banana, C = cherry) are chosen from (A, B, C), alongside their representation in the stars-and-bars method. A star (here, a letter) represents a chosen item, and a vertical bar “|” represents a separator between different fruit types.

Fruit multiset	Stars-and-bars diagram
AAA	AA I
AAB	AA B I
AAC	AA I C
ABB	A B B I
ABC	A B C I
ACC	A I C C
BBB	I B B B I
BBC	I B B C I
BCC	I B C C I
CCC	I I C C C I

In general, the formulas for the number of permutations, combinations, and multicombinations for choosing L items out of N distinct members are, respectively:

$$P(N, L) = \frac{N!}{(N-L)!},$$

$$C(N, L) = \binom{N}{L} = \frac{N!}{L!(N-L)!}, \quad (1)$$

$\left(\binom{N}{L}\right) = \binom{N+L-1}{L} = \frac{(N+L-1)!}{L!(N-1)!}$, where we adopt the notation $\left(\binom{N}{L}\right)$ (double braces) to denote the multicomination number, i.e., the number of size- L multisets from a set of N types.

While $P(N, L)$ and $C(N, L)$ are conceptually straightforward (they arise directly from the factorial formula and the binomial coefficient, e.g., via Pascal's triangle), the derivation of the multicomination formula is less immediate. One widely used technique to derive $\binom{N+L-1}{L}$ is the *stars-and-bars* method.^{1,2} In this method, one represents the L items to be chosen as L “stars” and divides them into N categories (representing the N

distinct types) using $N-1$ “bars”. The total number of symbols (stars + bars) in such a representation is $L+N-1$. By choosing which L of these $L+N-1$ positions are occupied by stars (equivalently, which $N-1$ positions are occupied by bars), one obtains the formula $\binom{N+L-1}{L}$. In essence, this counts the number of non-negative integer solutions to

$$a_1 + a_2 + \dots + a_N = L, \tag{2}$$

which is exactly the multicomination scenario where a_s is the number of times the s -th item is chosen (for $1 \leq s \leq N$).^{1,2}

For a concrete illustration, consider $N = 3$ kinds of fruit: apples (A), bananas (B), and cherries (C). If we multi-choose $L = 3$ pieces of fruit, there are $\binom{3+3-1}{3} = \binom{5}{3} = 10$ possible selections, corresponding to the number of multisets of size 3. Table 1 lists all 10 outcomes (left column) and shows their corresponding stars-and-bars representation (right column).

In this method, each chosen item is represented by a “star” (e.g., the letters A, B, and C for apples, bananas, and cherries), and vertical bars are inserted to divide the sequence into regions corresponding to each item type. For a selection of three fruit types, two bars are used to partition the stars into three sections. If no stars appear between two bars, that fruit type was not selected. For instance, the multiset $\{A, A, B\}$ —meaning two apples and one banana—is visually rendered as **AA|B|**, which corresponds equivalently to the star representation ****|*|**. Here, each letter (**A** or **B**) represents an identical star; the use of different letters is purely a visual aid. The interpretation in terms of apples, bananas, or other item types is determined entirely by the positions of the stars relative to the bars. In the present example, the first segment (before the first bar) contains two stars, indicating two apples; the middle segment contains one star, corresponding to a banana; and the final segment is empty, implying that no cherries were chosen. This diagrammatic approach establishes a precise one-to-one correspondence between all possible size- L multisets drawn from N types and all strings consisting of L stars and $N-1$ bars arranged in a row.

Mastery of these fundamental counting techniques is essential across a wide range of disciplines, including probability,³ statistics,⁴ computer science,⁵ and even quantum physics. Historically, the systematic study of combinations and binomial coefficients dates back centuries.² The formula for combinations with repetition, and the stars-and-bars argument above, became a standard part of combinatorics by the 19th--20th century.

Another powerful tool in combinatorics is the generating function technique. A generating function is a formal power series in which the coefficient of z^L represents the number of outcomes for a given L . Generating functions allow counting problems to be solved via algebraic methods.⁶ The use of generating functions in combinatorics dates back to Leonard Euler in the 18th century: Euler famously introduced infinite-product generating functions to study *integer partitions*, that is, representations of a positive integer as a sum of smaller positive integers in which the order of the summands does not matter.⁷ The associated *partition number* counts how many such decompositions exist. Euler’s insight was that

one can encode choices of parts in a partition by factors in a product; for example, he considered the infinite product $(1+x)(1+x^2)(1+x^3)\dots$ and observed how expanding it generates the partition numbers. In the context of combinations, generating functions provide an elegant way to derive counting formulas. We will next review the ordinary generating functions for combinations and multicombinations, before turning to our new interpretation in terms of *stairs*.

The generating function for combinations (without repetition) is obtained from the binomial expansion. Consider $(1+z)^N$: expanding this by the Binomial Theorem gives⁶

$$F_N(z) \equiv (1+z)^N = \sum_{L=0}^N \binom{N}{L} z^L, \tag{3}$$

since $(1+z)^N = \binom{N}{0}z^0 + \binom{N}{1}z^1 + \dots + \binom{N}{N}z^N$. This provides exactly the number of ways to choose L items from N without repetition. In this interpretation, in each factor of $(1+z)(z^0+z^1)$, the power z^1 or z^0 represents whether a given item is chosen or not, and the exponent of z in the product $(1+z)^N$ chosen counts the total number of chosen items.

For multicombinations, we allow an unlimited number of each item, implying that we need to use an infinite series. The geometric series⁶

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \tag{4}$$

represents choosing 0, 1, 2, 3, ..., of a given item. If we have N types of items available for repetition, the generating function for multicombinations is an infinite series:

$$B_N(z) \equiv \frac{1}{(1-z)^N} = \left(\frac{1}{1-z}\right)^N = \sum_{L=0}^{\infty} \binom{N}{L} z^L. \tag{5}$$

Indeed, multiplying N copies of the geometric series (4) corresponds to choosing some number a_1 of item 1, a_2 of item 2, ..., a_N of item N ; when we expand $\frac{1}{(1-z)^N}$, the coefficient of z^L in the expansion, it counts all possible ways to pick a total of L items with repetition from N available types, which are $\binom{N+L-1}{L}$. We can also express this formally as:

$$\frac{1}{(1-z)^N} = \sum_{a_1=0}^{\infty} \dots \sum_{a_N=0}^{\infty} z^{a_1+a_2+\dots+a_N} = \sum_{L=0}^{\infty} \sum_{\substack{a_1, \dots, a_N \geq 0 \\ a_1 + \dots + a_N = L}} z^L, \tag{6}$$

where in the double summation, we sum z^L over all N -tuples of nonnegative integers (a_1, a_2, \dots, a_n) that sums to L . The coefficient of z^L is therefore the number of solutions to $a_1 + \dots + a_n = L$, which matches the multicomination formula. In other words, we have shown combinatorially that

$$\binom{N+L-1}{L} = \sum_{\substack{a_1, \dots, a_N \geq 0 \\ a_1 + \dots + a_N = L}} 1, \tag{7}$$

reaffirming that $\binom{N+L-1}{L}$ counts the number of nonnegative solutions to (2).

The purpose of the present paper is twofold: (i) to reformulate the multicomination problem in terms of designing *non-decreasing stairs*; and (ii) to show how this point of view naturally leads to a two-variable generalization of the generating functions above, elaborating the connection between combinations with and without repetition. In what follows, we first introduce the stair interpretation and derive the cor-

responding results. Then we discuss how it connects to the classical methods (stars-and-bars) as well as some further implications.

■ Results and Discussion

Multicombination: A Staircase Analogy:

We recast the multicombination problem as follows: instead of thinking in terms of selecting L items out of N types, consider multi-choosing numbers from 1 up to N . That is, imagine we wish to pick L numbers from the set $1, 2, \dots, N$, with repetition allowed (equivalently, we wish to pick L not-necessarily-distinct numbers between 1 and N). For each $s \in 1, 2, \dots, N$, let a_s be the number of times s is chosen. If $a_s=0$, it means s is not chosen at all; if $a_s = 1$, s is chosen once; if $a_s = 2$, s is chosen twice; and so on. Clearly, equation (2) still holds in this scenario, which we recall here for convenience:

$$a_1 + a_2 + \dots + a_N = L. \tag{8}$$

This is just a rephrasing of the stars-and-bars description in terms of the counts a_s for each number.

Now, let us list the L chosen numbers in non-decreasing order, and call them α . By construction, we have

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_L, \quad \text{with } 1 \leq \alpha_1 \leq \alpha_L \leq N. \tag{9}$$

In fact, we can write the multiset explicitly in this sorted form as:

$$\underbrace{1, 1, \dots, 1}_{a_1 \text{ times}}, \underbrace{2, 2, \dots, 2}_{a_2 \text{ times}}, \dots, \underbrace{N, N, \dots, N}_{a_N \text{ times}} = (\alpha_1, \alpha_2, \dots, \alpha_L), \tag{10}$$

which makes it clear that $\alpha_1 \leq \dots \leq \alpha_L$ and that each α_k equals some s (with possible repetition according to a_s). The condition on successive α 's is $\alpha_{k+1} - \alpha_k \geq 0$ for $1 \leq k \leq L$. In this way, we obtain a non-decreasing sequence $\alpha = (\alpha_1, \dots, \alpha_L)$ with $1 \leq \alpha_k \leq N$ for all k .

We now map this α -sequence to another sequence $\beta = (\beta_1, \beta_2, \dots, \beta_L)$ of the same length L , defined by

$$\beta_k = \alpha_k + (k - 1) \quad \text{for each } k = 1, 2, \dots, L. \tag{11}$$

In other words, we add 0 to α_1 , 1 to α_2 , 2 to α_3 , and so on (adding $k-1$ to α_k). This transformation produces a strictly increasing sequence β with $\beta_1 < \beta_2 < \dots < \beta_L$. To see this, note that

$$\beta_{k+1} - \beta_k = (\alpha_{k+1} + k) - (\alpha_k + k - 1) = \alpha_{k+1} - \alpha_k + 1.$$

Since $\alpha_{k+1} \geq \alpha_k$, we have $\beta_{k+1} - \beta_k \geq 1$, so indeed $\beta_{k+1} > \beta_k$ for all k . Figure 1 illustrates an example.

What are the possible minimum and maximum values of the β 's? From the definition, $\beta_1 = \alpha_1 \geq 1$, so the minimum possible value of β_1 is 1 (achieved when $\alpha_1 = 1$). From

$$\beta_L = \alpha_L + (L - 1) \leq N + (L - 1), \tag{12}$$

the maximum possible value of β_L is $N+L-1$ (achieved when $\alpha_L = N$). Therefore, the β -sequence is a strictly increasing sequence of length L satisfying

$$1 \leq \beta_1 < \beta_2 < \dots < \beta_L \leq N + L - 1. \tag{13}$$

For example, the two extreme cases are:

$$\alpha = (1, 1, \dots, 1) \Rightarrow \beta = (1, 2, \dots, L),$$

and

$$\alpha = (N, N, \dots, N) \Rightarrow \beta = (N, N + 1, \dots, N + L - 1).$$

These show that β_1 can be as low as 1 and β_L as high as $N+L-1$.

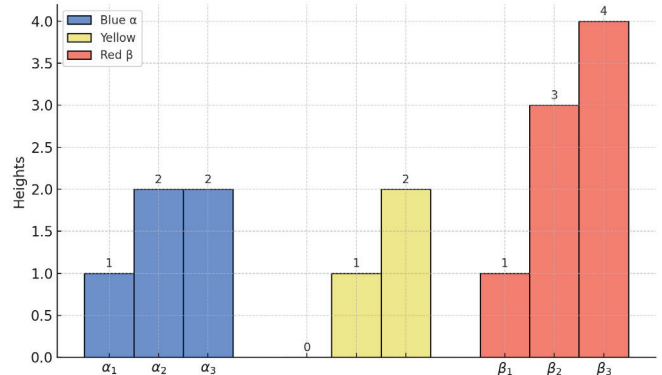


Figure 1: Example of the stair bijection: a non-decreasing α -stair (1,2,2) is mapped one-to-one to a strictly increasing β -stair (1,3,4) via the prescription in equation (11), $\beta_k = \alpha_k + (k-1)$ for $k=1, 2, 3$. Blue bars represent the α -sequence, red bars represent the β -sequence, and yellow bars indicate their difference $k-1$.

It is not hard to see that this $\alpha \mapsto \beta$ mapping is actually a bijection (one-to-one and onto) between the set of non-decreasing α -sequences in $[1, N]$ of length L and the set of strictly increasing β -sequences in $[1, N + L - 1]$ of length L . Indeed, given any strictly increasing sequence $1 \leq \beta_1 < \beta_2 < \dots < \beta_L \leq N + L - 1$ we can invert the relation (11) as

$$\alpha_k = \beta_k - (k - 1),$$

and check that $\alpha_{k+1} - \alpha_k = (\beta_{k+1} - \beta_k) - 1 \geq 0$, with $1 \leq \alpha_1 \leq \dots \leq \alpha_L \leq N$. This shows that every valid β -sequence corresponds to a unique non-decreasing α -sequence, and vice versa. Thus, we have established a bijection between multi-choose selections and ordinary combinations in a larger set.

What does this bijection mean combinatorially? It means that choosing L items from N with repetition (multicombinations) is in one-to-one correspondence with choosing L distinct items from a larger set of size $N + L - 1$ (ordinary combinations). The β -sequence can be interpreted as the set of L distinct numbers chosen from $1, 2, \dots, N + L - 1$, and clearly, there are $\binom{N+L-1}{L}$ such choices. Thus, we have derived the multicombination formula *bijectively*:

$$\left(\binom{N}{L} \right) = \binom{N+L-1}{L}, \tag{14}$$

recovering the result stated in equation (1). In this interpretation, one can imagine constructing a staircase diagram: the α -sequence $(\alpha_1, \dots, \alpha_L)$ can be seen as the heights of L steps (of equal width) in a non-decreasing “staircase,” whereas the β -sequence $(\beta_1, \dots, \beta_L)$ corresponds to a strictly increasing staircase profile of L steps (each subsequent step strictly higher than the previous one). We will not dwell on the geometric diagram here, but the idea is that each multiset selection corresponds to a unique set of step heights. Below, we will give an explicit example of this correspondence.

Two-variable generating functions for stairs:

The identification of multicombinations with the stair problem leads to a natural generalization of the generating function (5). We now introduce a *two-variable generating function* $B_N(z, x)$ for the stair interpretation:

$$B_N(z, x) := \prod_{s=1}^N \frac{1}{1 - zx^s}, \tag{15}$$

where z marks the stair length (number of items chosen) and x tracks the total height (sum of the values of the chosen items).

Expanding each factor as a geometric series, we get

$$B_N(z, x) = \prod_{s=1}^N \sum_{a_s=0}^{\infty} (zx^s)^{a_s} = \sum_{L=0}^{\infty} B_{N,L}(x) z^L,$$

for some coefficient polynomial $B_{N,L}(x)$ (the coefficient of z^L) depending on x . By collecting terms of the same power of z , one acquires

$$B_{N,L}(x) = \sum_{\substack{a_1, \dots, a_N \geq 0 \\ a_1 + \dots + a_N = L}} x^{1a_1 + 2a_2 + 3a_3 + \dots + Na_N}, \tag{16}$$

since each term of the expansion is of the form $z^{a_1 + \dots + a_N} x^{1a_1 + 2a_2 + \dots + Na_N}$, and we sum all contributions where the total exponent of z is L . The expression (16) has a clear meaning: we are summing $x^{\sum_{s=1}^N s \cdot a_s}$ over all multisets of size L , where $\sum_{s=1}^N s \cdot a_s$ is the sum of the values of the chosen numbers. But $\sum_{s=1}^N s \cdot a_s$ can be interpreted in terms of the α -sequence of the stair representation. For the expanded list (10), we have

$$\sum_{s=1}^N s a_s = \alpha_1 + \alpha_2 + \dots + \alpha_L, \tag{17}$$

which is exactly the sum of all step heights in the stairs. Thus, we can rewrite (16) as

$$B_{N,L}(x) = \sum_{1 \leq \alpha_1 \leq \dots \leq \alpha_L \leq N} x^{\alpha_1 + \alpha_2 + \dots + \alpha_L}, \tag{18}$$

where the sum runs over all non-decreasing sequences $(\alpha_1, \dots, \alpha_L)$ of length L with each $1 \leq \alpha_k \leq L$. In other words, $B_{N,L}(x)$ counts all non-decreasing stairs of length L by the total combined height of all steps (with step heights bounded between 1 and N). Setting $x=1$ in this formula, we recover the ordinary generating function for multicombinations, since $B_{N,L}(1) = \sum_{\alpha} 1 = \binom{N+L-1}{L}$. Indeed, one can see directly that

$$B_N(z, 1) = \prod_{s=1}^N \frac{1}{1-z} = \frac{1}{(1-z)^N} = B_N(z), \tag{19}$$

consistent with our earlier definition of $B_N(z)$ in equation (5). We have used the notation B to emphasize the analogy with bosonic counting (i.e., combinations with repetition). Different arguments indicate standard specializations of the same generating function: $B_N(z)$ denotes the ordinary generating function, $B_N(z, x)$ its two-variable refinement that tracks total height, and $B_{N,L}(z)$ the coefficient of z^L in this expansion. An analogous notation will be introduced below for combinations without repetition, using the letter F (for fermion). In parallel, for ordinary combinations (without repetition), one can define an analogous two-variable generating function:

$$F_N(z, x) := \prod_{t=1}^N (1 + zx^t) = \sum_{L=0}^N F_{N,L}(x) z^L, \tag{20}$$

where

$$F_{N,L}(x) = \sum_{\substack{b_1, \dots, b_N \in \{0,1\} \\ b_1 + \dots + b_N = L}} x^{1b_1 + 2b_2 + 3b_3 + \dots + Nb_N}. \tag{21}$$

Here $b_t \in \{0,1\}$ indicates whether the item t is chosen ($b_t=1$) or not ($b_t=0$). The condition $b_1 + \dots + b_N = L$ means exactly distinct items are chosen. Thus $F_{N,L}(x)$ encodes the total ‘‘height’’ (sum of values) of all choices of L distinct items from $1, \dots, N$. In fact, if the chosen set of L distinct numbers are written in increasing order $(\beta_1 < \beta_2 < \dots < \beta_L)$, then one can verify

$$\sum_{t=1}^N t b_t = \beta_1 + \beta_2 + \dots + \beta_L. \tag{22}$$

Thus, we can rewrite (21) as

$$F_{N,L}(x) = \sum_{1 \leq \beta_1 < \beta_2 < \dots < \beta_L \leq N} x^{\beta_1 + \beta_2 + \dots + \beta_L}. \tag{23}$$

When $x=1$, $F_{N,L}(1) = \binom{N}{L}$ (the number of ways to choose L distinct items from N), and indeed $F_N(z, 1) = (1+z)^N = F_N(z)$ as expected.

The generating function $F_N(z, x)$ itself is standard in the theory of generating functions and statistical mechanics,^{6,8-10} the novelty here lies in its interpretation in terms of stair sequences and its explicit bijective relation to multicombinations.

Now, an interesting relationship emerges between $B_N(z, x)$ and $F_N(z, x)$. Consider the strictly increasing β -sequence corresponding to a given non-decreasing α -sequence via our stair bijection. For a fixed L , suppose $\alpha = (\alpha_1, \dots, \alpha_L)$ maps to $\beta = (\beta_1, \dots, \beta_L)$ by the relation $\beta_k = \alpha_k + k - 1$. We can relate the sums of α 's and β 's (i.e., (17) and (22)) as follows:

$$\sum_{k=1}^L \beta_k = \sum_{k=1}^L (\alpha_k + k - 1) = \left(\sum_{k=1}^L \alpha_k \right) + \frac{1}{2} L(L-1). \tag{24}$$

Here, the second term $\frac{L(L-1)}{2}$ comes from $0+1+2+\dots+(L-1)$. Now, if we consider a combination scenario with N replaced by $N' = N + L - 1$, we note that any strictly increasing sequence $(\beta_1, \dots, \beta_L)$ with $\beta_1 > 1$ and $\beta_L \leq N + L - 1$ can be viewed as a choice of L distinct numbers from $1, 2, \dots, N + L - 1$. In fact, as shown earlier, any α -sequence for (N, L) corresponds bijectively to a β -sequence for $(N + L - 1, L)$. Using the relation (24), we can now connect the two-variable generating functions:

$$F_{N+L-1,L}(x) = x^{\frac{L(L-1)}{2}} B_{N,L}(x). \tag{25}$$

In words: the sum of $x^{\text{(total height of strictly increasing stair)}}$ over all strictly increasing stairs of length L up to height N (which is $F_{N,L}(x)$) is related to the sum of $x^{\text{(total height of non-decreasing stair)}}$ over all non-decreasing stairs of length L up to height N (which is $B_{N,L}(x)$), by a factor of $x^{L(L-1)/2}$ ---after adjusting N to $N+L-1$. Here, the term total height refers to the sum of the heights of all steps in the staircase, rather than the height of the highest step. Equivalently, the total height is the area under the staircase profile. Substituting the relation (25) back into the generating functions (16) and (21) confirms the identity (25). In particular, when $x=1$, the identity (25) reduces to the multicomination formula, $\binom{N+L-1}{L} = \binom{N}{L}$ as given in equation (14). For example, for $N=2$ and $L=2$, we have

$$B_{2,2}(x) = x^2 + x^3 + x^4, \quad F_{3,2}(x) = x^3 + x^4 + x^5 = x B_{2,2}(x), \tag{26}$$

which indeed reflects the relation $F_{2+2-1,2}(x) = x^1 B_{2,2}(x)$ in this case (since $L(L-1)/2 = 1$ when $L=2$).

Correspondence with the stars-and-bars method:

To illustrate the stair interpretation concretely, let us revisit the fruit example in Table 1 and map each multicomposition to its corresponding stair sequences.

Table 2 shows the detailed one-to-one correspondences between each fruit multiset, the non-decreasing α -sequence, the strictly increasing β -sequence, and the stars-and-bars representations (both for the β -sequence and for the original fruit

labels). We use the numeric values 1,2,3 for A, B, C, respectively, when writing the α and β sequences.

Table 2: Each row maps a multichoose outcome to its corresponding stair sequences and equivalent stars-and-bars encodings for $N=3$ and $L=3$. The α -sequence is the non-decreasing listing of the chosen item values (using 1 for A, 2 for B, 3 for C). The β -sequence is obtained by adding 0, 1, 2 to the α terms, yielding a strictly increasing sequence between 1 and 5. The “ β & bars” column shows the positions (1 through 5) of the chosen β 's as stars in a stars-and-bars diagram (with two bars for 3 bins), and the last column is the original stars-and-bars diagram for the fruits.

Fruit multiset	α -sequence	β -sequence	β & bars	Fruit & bars
AAA	111	123	123	AA I
AAB	112	124	12 4	AA B
AAC	113	125	12 5	AA C
AAB	122	134	1 34	A BB
ABC	123	135	1 3 5	A B C
ACC	133	145	1 45	A I CC
BBB	222	234	234	BBB
BBC	223	235	23 5	B B C
BCC	233	245	2 45	B C C
CCC	333	345	345	C C C

For instance, the selection AAB corresponds to $\alpha = (1, 1, 2)$ (since in sorted order it consists of two 1's and one 2). This yields $\beta = (\alpha_1, \alpha_2 + 1, \alpha_3 + 2) = (1, 2, 4)$. The β -sequence (1, 2, 4) can be interpreted as choosing $\{1, 2, 4\}$ out of $\{1, 2, 3, 4, 5\}$ (since $N+L-1=5$ for this case). The “ β & bars” column in Table 2 shows the stars-and-bars diagram corresponding to this β representation: for AAB, choosing $\{1, 2, 4\}$ out of 5 means positions 1 and 2 are stars, position 3 is a bar, position 4 is a star, and position 5 is a bar, giving the string $12|4|$. Finally, replacing the numbers 1, 2, 4 with the corresponding fruit letters in order gives the original fruit diagram $AA|B|$. This confirms that our stair interpretation is in perfect correspondence with the classical stars-and-bars approach. Table 2 explicitly demonstrates the bijection between the fruit multiset, its α -sequence, β -sequence, and the two equivalent stars-and-bars representations.

Essentially, the β -representation provides another viewpoint on the standard stars-and-bars bijection. The chosen β positions correspond to picking which L of the $N+L-1$ slots contain stars in the stars-and-bars model.

Discussion:

The two-variable generating functions (15) and (20) offer a refined method for combinatoric enumeration: not only do they distinguish the number of items, but they also track the sum of the values (heights) selected. The identity (25) we derived shows a precise relationship between these refined counts for combinations with and without repetition. By considering an arbitrarily large supply of item types, one finds that the two sequences coincide up to a simple factor. In fact, if is very large compared to , choosing with repetition or without repetition becomes effectively the same for the first positive integers. More formally, in the limit we acquire

$$\lim_{N \rightarrow \infty} B_{N,L}(x) = \prod_{s=1}^L \frac{x}{1-x^s} = \frac{x^L}{\prod_{s=1}^L (1-x^s)},$$

$$\lim_{N \rightarrow \infty} F_{N,L}(x) = \prod_{t=1}^L \frac{x^t}{1-x^t} = \frac{x^{\frac{1}{2}L(L+1)}}{\prod_{t=1}^L (1-x^t)}, \quad (27)$$

using the formula for geometric series and noting that $F_{N,L}(x)$ for infinite N would include all L distinct positive integers (hence the product from $t=1$ to L). The higher powers of the denominators arise from the contributions of each possible step height in the staircase interpretation: in the infinite- N limit each factor $(1-x^t)^{-1}$ corresponds to allowing arbitrarily many steps of height t . We see that $\lim_{N \rightarrow \infty} F_{N,L}(x)$ equals $\lim_{N \rightarrow \infty} B_{N,L}(x)$ times a factor $x^{\frac{L(L-1)}{2}}$. This coincides with our earlier identity (25) in the special case $N = \infty$.

Interestingly, this type of generating function identity is known in a physics context: in quantum mechanics, *bosons* correspond to combinations with repetition and *fermions* to combinations without repetition. The generating function $\prod_{s=1}^{\infty} \frac{1}{1-zx^s}$ is the grand canonical partition function for bosons, and $\prod_{s=1}^{\infty} (1+zx^s)$ is that for fermions. In the limit of infinitely many states for particles confined to a one-dimensional harmonic potential, the two partition functions coincide up to the overall factor discussed above, reflecting a combinatorial identity recognized by physicists.⁸⁻¹⁰

Finally, beyond the combinatorial insights, the stair interpretation provides a more visual way for students to understand combination problems. By translating an abstract stars-and-bars configuration into a step diagram, one can literally “see” a combination with repetition as building a staircase of a certain length and bounded height. This perspective might be useful in discovering new combinatorial identities or generalizations, as we demonstrated with generating functions.

Conclusion

We have introduced a bijective stair interpretation for multicombinations in terms of non-decreasing and strictly increasing sequences (viewed as stairs), and we have derived a generating function identity connecting combinations with and without repetition.

This viewpoint offers an intuitive bijective proof of the formula for combinations with repetition and elucidates the relationship between these and ordinary combinations. We compared the stair method with the traditional stars-and-bars method and found them to be in perfect correspondence, as illustrated in Table 2. Additionally, by incorporating a weight variable x to track the sum of chosen values, we developed two-variable generating functions $B_N(z,x)$ and $F_N(z,x)$ that unify combinations with and without repetition in a single framework. The resulting identity (25) relating $B_{N,L}(x)$ and $F_{N+L-1,L}(x)$ provides a deeper algebraic insight into this connection and links our combinatorial findings to known results in both combinatorics and quantum physics.

We also provided historical context and references to help students appreciate the development of these ideas. In summary, this paper demonstrates how a fresh combinatorial interpretation (the “stairs”) can lead to elegant proofs and generalizations of standard counting formulas. The stair viewpoint may, in turn, inspire further generalizations in other combinatorial structures, such as integer partitions¹¹ or Young tableaux.¹²

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