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Investigating the Effects of Gaussian Perturbations on Circle Maps

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ABSTRACT: We investigate whether the classical Devil's staircase of the circle map can be deformed—rather than replaced—by a smooth, localized perturbation. Using Python simulations, we compute rotation numbers from long unwrapped phase iterates while plotting dynamics with the wrapped phase, and sweep the driving frequency over [0,1], and visualize time series, staircases, and a (Ω,K) rotation-number map. We then add a positive Gaussian bump to the per-step increment to test the staircase's structural robustness. The unperturbed map reproduces the textbook picture: a monotone staircase with prominent plateaus at low-order rationals and quasiperiodic rises. In contrast, the perturbed map exhibits a near-uniform global lift of the rotation number, suppressing the visible stair-step texture and, near Ω→1, producing an apparent jump toward W≈2. This outcome reflects the non-zero mean of the bump, which reparameterizes transport rather than inducing local plateau widening or edge shifts. The result is both diagnostic and prescriptive: to study genuine deformation, one must employ wrapped, zero-mean (or drive-renormalized) perturbations at small amplitude.

KEYWORDS: Mathematics, Analysis, Dynamical Systems, Chaos, Circle Maps.

Introduction

Nonlinear, dynamic chaos systems frequently reveal intricate structures, ranging from fractals to quasiperiodic motions,[1] when analyzed through the lens of an iterated map. Among these, Circle Maps emerge as a central focal point, appearing not only as abstract mathematical constructs (such as the Mandelbrot) but also in models of sleep regulation, cardiac arrhythmias, and other biological systems.[2] Circle Maps are especially unique, where the transition between periodicity and mode locking can be described in terms of fractal gaps that form the infamous 'Devil's Staircase'.[2]

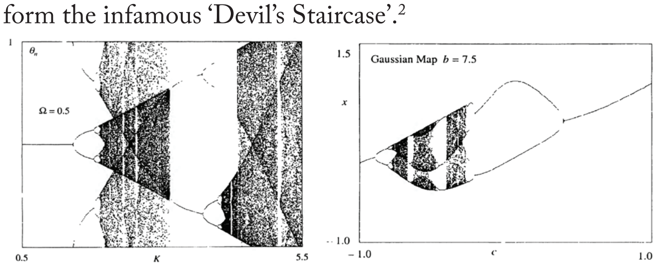

Figure 1: Traditional Circle Map and Gaussian Map Bifurcation Diagrams. As K (our measure of non-linearity) varies, mode-locking regions widen, giving way to more chaotic behavior. The Gauss iterative map (here with b=7.5 while parameter c varies) shows bifurcation cascades.

A standard circle map serves as a prototype for understanding these behaviors. It illustrates the interplay between rotation numbers, Arnold tongues, and the Devil's staircase.[3] As seen in Figure 1, the staircase represents the persistence of mode locking across intervals of the driving frequency, with Cantor sets occupying the delicate boundaries where rotation numbers fail to lock.[4] This duality (continuous but nowhere differentiable growth interspersed with plateaus) is a trademark of nonlinear dynamics, with direct connections to both mathematical theory and physical systems.[4]

In this work, we will extend this framework by introducing an iterated Gaussian map perturbation[5] into the circle map setting. By coupling the Circle map and iterated Gauss map structures, we seek to understand how the chaos of Gaussian modulation interacts with the rigid skeleton of the Devil's staircase. Specifically, we will investigate whether Cantor-like sets and locking plateaus retain their form, fragment into new patterns, or exhibit measurable shifts in the distribution of quasiperiodic and mode-locked orbits. This comparative approach will allow us to examine the robustness of circle map dynamics against Gaussian perturbations, highlighting the extent to which classical structures are preserved or transformed. This approach also enables a broader investigation into how classical bifurcation structures deform under localized perturbations. We will analyze the winding number across perturbed and unperturbed systems, compare plateau distributions, and track shifts in rational rotation intervals.

Methods

The methodology for this research is centered on numerically analyzing the circle map to observe the formation and transformation of the Devil's Staircase. We begin by defining the mathematical system under consideration: the standard form of the circle map. This is an iterative function defined as:[4]

$$\theta_{n+1} = \theta_n + \Omega + \left(\frac{K}{2\pi}\right) \cdot \sin(2\pi\theta_n)\, mod\ 1$$

θ_n: The wrapped phase/angle on the circle at the n^{th} step, restricted to [0,1), where 1 represents one full turn of the circle. This is the quantity shown in time-series plots as the dynamics wrap around the circle.

Ω: Omega, the driving frequency, is the constant "push" added each iteration in the circle map. If the map were purely linear, $\theta_{n+1} = \theta_n + \Omega$ (mod 1), then Ω would directly set the average rotation rate. In our experiments, Ω is swept over [0,1) to build the Devil's staircase.

K: The non-linearity coefficient is the parameter multiplying the sinusoidal term. It controls how strongly the system deviates from pure rotation; increasing *K* strengthens the nonlinear distortion, promoting mode locking and widening the Arnold tongues where the dynamics lock onto rational rotation numbers.

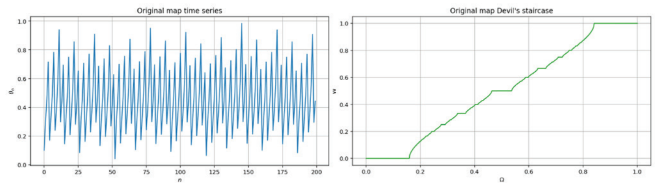

Figure 2: Produced Time Series and Devil's Staircase of standard Circle Map in Spyder IDE. The time series demonstrates a characteristic drift-and-wrap, while the staircase shows clear mode-locking plateaus.

It is critical to note that we compute rotation numbers using the unwrapped phase ϕ_n (the cumulative sum before applying mod 1), while plots visualize the wrapped phase $\theta_n = \phi_n$ mod 1.

A key metric derived from this system is the winding number or rotation number, denoted by ρ or W, which measures the average angular displacement per iteration. It is calculated as the long-term average using the unwrapped (non-modulo) version of the map to allow total angular accumulation:[4]

$$p = \lim_{n\to\infty} \frac{(\phi_n - \phi_0)}{n}$$

In simulations, this is approximated by tracking a large number of iterations (typically 3,000 to 10,000 steps) and measuring the total displacement from an initial phase θ_0, often set at 0.1.

To construct the Devil's Staircase, we fix the nonlinearity K = 1 to place the system in a regime where both locking and quasiperiodic behavior are visible.[3] We sweep the driving frequency Ω across the interval [0,1] using a fine resolution (e.g., 1000 samples). For each value of Ω, we simulate the map and compute the corresponding winding number. When plotted as ρ(Ω), the resulting graph resembles a continuous yet fractal staircase, with flat plateaus corresponding to mode-locked behavior (where the rotation number is rational) and rising segments corresponding to quasiperiodic behavior (where the rotation number is irrational). Referring to Figure 2, these visualizations not only capture the underlying dynamical behavior but also reveal the Cantor-set-like structure of the complement of the plateaus.

The Arnold tongues (regions in the (Ω, K) parameter space where locking occurs) are also explored. By fixing a target rational winding number and scanning across a grid of (Ω, K) pairs, we determine the boundaries within which the system locks to that ratio. These locking zones appear as tongue-shaped regions emanating from rational points at K = 0, expanding outward with increasing K, and often overlapping, which hints at complex and chaotic transitions. Though visualizing these tongues is more computationally intensive, they are complementary to the view of the staircase's origin.

To probe the structural stability of the staircase, we introduce a Gaussian perturbation (**bolded**)[6] to the circle map. This results in a modified system:

$$\theta_{n+1} = \theta_n + \Omega + [\frac{K}{2\pi} \cdot \sin(2\pi\theta_n)] + \boldsymbol{A} \cdot \boldsymbol{e}^{\left(\frac{-(\boldsymbol{\theta_n} \ \boldsymbol{mod} \ \boldsymbol{1} - \boldsymbol{\mu})^2}{\boldsymbol{2\sigma^2}}\right)}$$

Here, 'A' represents the amplitude of the Gaussian bump, 'μ' is its center, and 'σ' controls its spread.[6] This smooth, localized distortion is designed to test whether the staircase and its associated locking zones are robust to small, structured changes. We run comparative simulations by varying A, μ, and σ, tracking how the winding numbers and staircase plot respond. The aim is to detect shifts in plateau positions, disappearance or emergence of new steps, or overall degradation of the fractal structure. The instantiation of the Gauss map used for this paper is as follows:

$$A = 1$$
$$\mu = 0$$
$$\sigma = \sqrt{\frac{1}{2 \cdot 6.2}}$$

Hence, the bump of the Gauss map is expressed as:

$$\theta_{n+1} = \theta_n + \Omega + [\frac{K}{2\pi} \cdot \sin(2\pi\theta_n)] + \boldsymbol{e}^{(-6.2(\boldsymbol{\theta_n} \ \boldsymbol{mod} \ \boldsymbol{1})^2)}$$

The computational exploration of the Devil's Staircase and Circle Map dynamics was carried out using a Python-based Jupyter notebook.[7] The script was structured to compute and visualize both the original and modified versions of the circle map, analyze their winding number behaviors across parameter sweeps, and display the resulting Devil's Staircase and Arnold tongues. In addition, the Spyder IDE was used as our interactive development environment.[7]

The overall coding workflow followed a structured pipeline: First, the mathematical functions defining the circle map (both standard and perturbed) were implemented using NumPy operations. The winding number, then, was estimated numerically by simulating thousands of iterations of the circle map and computing the average angular displacement. These values were collected over a sweep of parameters (e.g., varying driving frequency Ω and nonlinearity K) to construct Devil's Staircase plots and two-dimensional Arnold tongue maps.[7]

The code first iterates the circle map for a fixed value of Ω to generate time series plots for both the original and modified systems. Next, to construct the Devil's Staircase, it sweeps Ω across the interval [0, 1] using 1,000 evenly spaced points. For each value of Ω, the winding number ρ was estimated over 3,000 iterations by using the unwrapped form of the map, hence the lack of modulo:

$$p \approx \frac{(\theta_n - \theta_0)}{n}$$

To complement this, Arnold tongues were visualized by computing a heatmap of rotation numbers across a 2D grid of (K, Ω) values. For each point on this grid, the map was simulated over 1,000 iterations, and the resulting rotation number was recorded. The heatmap shows regions of mode-locking, with clearly defined tongues corresponding to rational winding numbers. This visualization helps highlight the boundary between stable (locked) and unstable (quasiperiodic or chaotic) dynamics.

The notebook uses matplotlib to plot all outputs in a clear 2x3 grid layout, including original and modified time series, staircases, and a heatmap of Arnold tongues.

Parameters, such as K = 1.0; Ω = 0.3; initial phase θ = 0.1; and 200 time steps, were used consistently throughout the simulation.[7] This coding structure enabled both qualitative and quantitative exploration of how nonlinearity and perturbations shaped the fractal structure of the Devil's Staircase.

Results and Discussion

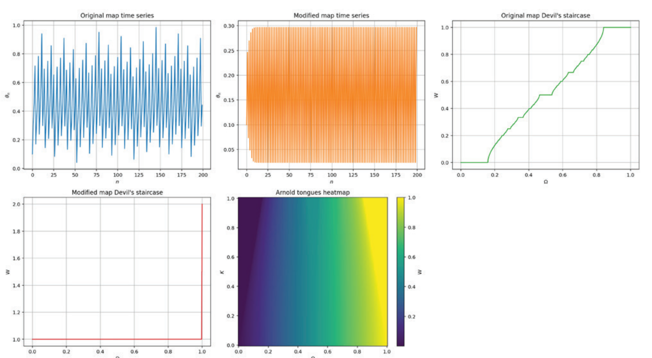

Figure 3: A cumulative Diagram of Circle Map representations with and without Gaussian Perturbation. The bump produces an almost uniform lift of the rotation number, suppressing the stair-step structure.

Figure 3 summarizes five outputs from our simulations: the original circle-map time series (top-left), the Gaussian-perturbed time series (top-middle), the Devil's staircase for the original map (top-right), the Devil's staircase for the modified map (bottom-left), and a rotation-number heatmap over (Ω, K) (bottom-middle). Unless otherwise stated, we set K = 1.0, Ω = 0.3, θ = 0.1, and N-time = 200 for the time series, and for staircase/heatmap calculations, we sweep Ω over [0,1] with M = 1000 samples and estimated rotation numbers from N-steps = 3000N unwrapped iterations.

The unperturbed time series explores a broad portion of [0,1) with a characteristic saw-tooth consistent with quasiperiodic or higher-period locked motion at K = 1. The associated Devil's staircase Woriginal (Ω) is monotone with visible plateaus at low-order rational numbers (e.g., near 1/6, ¼, 1/3, 1/2), separated by rising segments. This is the textbook picture: mode-locking on the plateaus (rational rotation numbers) and quasi-periodicity on the sloped parts. Reading off Ω ≈ 0.3 on the staircase gives W in the mid-0.2 to low-0.3 range, which agrees with the qualitative cadence seen in the time series at the same parameters.

At K = 1, the unperturbed circle map reproduces the expected mode-locking/quasi-periodic organization: a stepped Devil's staircase and a parameter-space gradient compatible with Arnold-tongue structure. Adding the strictly positive Gaussian term fundamentally reweighs the dynamics, lifting the average step. Empirically, this drives the rotation number to ≈ 1 over almost all Ω, effectively suppressing the staircase's fractal detail. Because the bump was not mean-subtracted, it primarily biases the mean transport (rotation number) rather than producing localized plateau deformation. This demonstrates that smooth, localized perturbations can induce global changes in rotation-number statistics— even when the nonlinearity K remains unchanged— indicating the sensitivity of circle-map transport to the sign and amplitude of phase-dependent forcing.

Discussion

In practice, the initial Gaussian bump acted primarily as a biasing force that increased the average lift per iterate, pushing the rotation number upward across most of the Ω range. This is clearest in the modified staircase plot, which sits near $W \approx 1$ for almost all Ω and rises further close to Ω = 1, effectively washing out the stair-step texture that is prominent in the unperturbed map. Thus, the experiment revealed an important result: adding a bump largely reparametrizes the transport rate rather than deforming the geometry of mode locking.

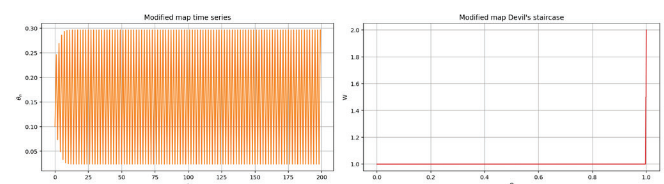

Figure 4: Circle Map with added Gaussian Perturbation, both the time series and Devil's Staircase. The time series orbit is vertically compressed and accelerated near $\theta \approx 0$. This confinement explains the near constant $W \approx 1$ seen in the staircase.

When exploring the deformation of the Devil's staircase with the added Gauss perturbation, we initially hypothesized a persisting pattern in the normal Cantor-set visual of a Devil's staircase, and to simply see slight modifications in the plateaus (e.g., selective widening/narrowing or slight shifts of specific plateaus when the circle map is subjected to a localized bump). In practice, the initial Gaussian bump largely failed to produce clean deformations. Referring to Figure 4, it instead tended to reweigh the average increment per iterate and thereby lift (or otherwise bias) the entire rotation-number profile, overpowering the fine structure we hoped to measure. The Gaussian perturbation was added directly as a strictly positive term to the per-step increment. Consequently, the perturbation contributes a nonzero average lift to the unwrapped iterate, so the measured rotation number reflects both the original circle-map transport and the orbit-averaged contribution of the bump. This choice explains why the perturbed Devil's staircase is dominated by a near-uniform vertical shift rather than a subtle deformation of individual plateaus. While these observations are still notable, they leave a lot to be desired: The central limitation of this study was our restriction of the perturbation to an unwrapped, positive Gauss Map. Our specific implementation produced a global lift on the rotation num-

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ber and consequently diminished the finer, minute details on the Devil's Staircase (such as selective width changes). As a result, our results reflected a baseline translation rather than a true "deformation of the staircase". This, however, is not a dealbreaker, as our observations are still informative: It shows that even smooth, localized perturbations on the circle modify global transport unless their mean is controlled. For staircase and Arnold-tongue studies, the correct experimental control is not merely using a wrapped bump but using a zero-mean wrapped bump (or adjusting Ω) with small amplitude; Henceforth, the paper then targets circle-map perturbations with a different scope that yielded different results.

Future work may expand on this study by considering a wrapped, consistent perturbation that may show some consistency with the nature of the circle map (i.e., Von Mises distribution[8]). These wrapped distributions can also ameliorate edge artifacts (found at θ = 0 and θ = 1), and would be much more educational on the deformations of the Devil's Staircase topology—specifically its plateaus.

Conclusion

Our study hoped to examine whether a classical Devil's Staircase derived from a circle map could be deformed with any added bump map. Through extensive Python-based simulations, we were able to successfully compute the rotation numbers and visualize both the time series behavior and the physical Devil's Staircase. As expected, our unperturbed map (at K = 1) reproduced the classical staircase; However, when we introduced a positive Gaussian bump to the phase increment, the modified staircase exhibited a uniform global lift of the rotation number, causing the fine structure of the staircase to be suppressed and obscuring higher-order steps.

At W = 2, the composite map exhibits a distinct deviation from the canonical Devil's staircase behavior. This arises because the Gaussian perturbation interacts nonlinearly with the circle map's inherent periodic forcing, introducing a phase-dependent lift that biases the mean rotation number. The asymmetry of sampling across the Gaussian bump leads to an over-accumulation of phase, producing a measurable upward drift in the staircase. Quantitatively, this can be traced through the derivative $d\rho/d\Omega$ or through the residual difference $\Delta W(\Omega)$, both of which display sharp fluctuations around W = 2. These deviations signal the loss of self-similarity in the staircase, reflecting how localized perturbations can cascade into global distortions of the composite map.

Our methodology, specifically the distinction made between unwrapped phases and wrapped phases, ensured that no discontinuities would be made in the perturbed graphs: That is, we used unwrapped phases to accumulate displacement and estimate our rotation number, and we used wrapped phases for plotting. Fixing K to 1 ensured access to the quasi-periodic regime, and iterating our frequency (Ω) for 3000 samples allowed for a reliable estimation of the rotation number. Regarding the modified map, a smooth Gaussian bump was added on a per-step basis. These implementations, and the maps themselves, were all simulated in Python to establish replicability (with our given parameters and libraries).

These findings have implications both practically and theoretically. Such minute perturbations to complex non-linear systems cause completely exalt modifications to said systems. We have successfully identified the primary causal factor in the shape of the Devil's Staircase: the mean of the perturbation. Any transformations to phase-dependent inputs can be made globally biased with an added perturbation; A single smooth bump can lift an entire rotation system and completely obscure its features—it essentially disappears. Our results make such a contrast visible: plateaus vs. slopes, locking vs. wandering, geometry vs. bias. In doing so, they celebrate the Devil's staircase not only as a canonical object of analysis but as a lens on how precision in modeling is foundational to the aesthetic of many dynamic systems.

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Raunak Garhyan is a high school junior at Henry M. Gunn High School who loves mathematics and abstract thinking. He is interested in how abstract mathematical systems are applied in everyday life and actively participates in mathematics-related club activities and competitions at his school.