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A Visual Derivation Using a Signed Pascal's Triangle for Expansions of sin(nθ) and cos(nθ)

Sihwan Lee

Hong Kong International School, 1 Red Hill Road, Tai Tam, Hong Kong (China), 00000; soohak280646@gmail.com

ABSTRACT: This paper presents a signed version of Pascal's triangle as a visual method to organize expansions of $sin(n\theta)$ and $cos(n\theta)$. Using these expansions, we compare $sin(n\theta)+cos(n\theta)$ to $(sin\theta+cos\theta)^n$. Although these identities can be derived from De Moivre's theorem, the main purpose is to provide a visual aid in understanding these expansions.
KEYWORDS: Mathematics, Pascal's Triangle, De Moivre's Theorem.

Introduction

In this paper, we investigated the expression $(\cos\theta+\sin\theta)^n$ and examined how it relates to $cos(n\theta)$ and $sin(n\theta)$. This expression is reminiscent of De Moivre's Theorem, which involves complex numbers through $(cos\theta+i\cdot sin\theta)^n= cos(n\theta)+i\cdot sin(n\theta)$. However, we focused on the real number version, which is without i, and uncover their identity.

It is already known that $cos(n\theta)$ and $sin(n\theta)$ can be written as polynomials using $cos\theta$ and $sin\theta$.[1-4] These results usually arise from De Moivre's Theorem or recursive trigonometric formulas.[1,2] But instead of focusing on algebraic derivations, we wanted to find a way to visually reflect those identities using combinatorial structure, specifically, through a variation of Pascal's Triangle.

Pascal's Triangle is a well-known structure in mathematics, where each row represents binomial coefficients arranged in a triangular format. While its applications in algebra are widely understood, its connection to trigonometry is not as well explored as algebra.[3,4]

To do this, we developed a modified version of Pascal's Triangle where, letting the leftmost coefficient in each row be coefficient 1 and counting left to right, the coefficients with numbers equivalent to 0,3 (mod 4) in each row are assigned negative signs. For example, the third row is [1,2-1], and the fourth row is [1,3,-3,-1].

1
1 1
1 2 −1
1 3 −3 −1
1 4 −6 −4 1
1 5 −10 −10 5 1
1 6 −15 −20 15 6 −1
1 7 −21 −35 35 21 −7 −1
1 8 −28 −56 70 56 −28 −8 1
1 9 −36 −84 126 126 −84 −36 9 1
1 10 −45 −120 210 252 −210 −120 45 10 −1

When we applied this modified triangle, in every row, a summation using the odd terms as coefficients summed to $cos(n\theta)$, and a similar summation using the even terms summed to $sin(n\theta)$:

$$\cos(n\theta)=p_{n-1,1}c^n+p_{n-1,3}c^{n-2}s^2+p_{n-1,5}c^{n-4}s^4+p_{n-1,7}c^{n-6}s^6+\cdots$$

$$\sin(n\theta)=p_{n-1,2}c^{n-1}s^1+p_{n-1,4}\cos^{n-3}s^3+p_{n-1,6}\cos^{n-5}s^5+p_{n-1,8}\cos^{n-7}s^7+\cdots$$

Where $c=cos(\theta)$, $s=sin(\theta)$, and $P_{a,b}$ denotes the b-th entry in the a-th row of the modified Pascal's triangle.

Note that while the topmost row of the triangle is the 0th row, the leftmost term of each row is the first term. This choice is to aid in intuitivity in reading summations, where a summation with n terms contains terms 1 through n rather than 0 through $n-1$.

By using two different versions of Pascal's Triangle, namely the standard triangle for expanding $(cos\theta+sin\theta)^n$ and the modified triangle for reconstructing $cos\theta+sin\theta$, we were able to establish an identity for the difference between the expressions.

Also, the modified triangle allowed us to generalize the trigonometric sums $\Sigma_{n=1}^{k}\cos(n\theta)$ and $\Sigma_{n=1}^{k}\sin(n\theta)$ in terms of $cos\theta$ and $sin\theta$, respectively.

Results

Theorem 1.

Letting the terms of the -th row be $[p_{n,1},p_{n,2},p_{n,3},\cdots p_{n,n+1}]$, $\cos(n\theta)=p_{n,1}\cos^n(\theta)+p_{n,3}\cos^{n-2}(\theta)\sin^2(\theta)+p_{n,5}\cos^{n-4}(\theta)\sin^4(\theta)+p_{n,7}\cos^{n-6}(\theta)\sin^6(\theta)\ldots$ and $\sin(n\theta)=p_{n,2}\cos^{n-1}(\theta)\sin^1(\theta)+p_{n,4}\cos^{n-3}(\theta)\sin^3(\theta)+p_{n,6}\cos^{n-5}(\theta)\sin^5(\theta)+p_{n,8}\cos^{n-7}(\theta)\sin^7(\theta)\ldots$.

Proof.

We will derive formulas for $\cos(n\theta)$ and $\sin(n\theta)$ using De Moivre's Theorem, and then prove that the terms of the formulas are the same as row n of the modified Pascal's triangle.

First, recall De Moivre's Theorem:[1,2]

De Moivre's Theorem states that $(cos(\theta)+i\cdot sin(\theta))^n = cos(n\theta)+i\cdot sin(n\theta)$. Continuing to use the notation of $cos(\theta)=c$ and $sin(\theta)= s$, expanding the left-hand side gives:

$$\binom{n}{0}c^n+\binom{n}{1}c^{n-1}i^1s^1+\binom{n}{2}c^{n-2}i^2s^2+\binom{n}{3}c^{n-3}i^3s^3+\cdots=\cos(n\theta)+i\sin(n\theta).$$

Simplifying the powers of i and separating the real and imaginary parts of the equation, we get two somewhat simpler equations:
$\binom{n}{0}c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 - \binom{n}{6}c^{n-6}s^6 \cdots = \cos(n\theta)$ and
$\binom{n}{1}c^{n-1}s^1 - \binom{n}{3}c^{n-3}s^3 + \binom{n}{5}c^{n-5}s^5 - \binom{n}{7}c^{n-7}s^7 \cdots = \sin(n\theta)$.

Thus, we have explicit binomial expansions for both $\cos(n\theta)$ and $\sin(n\theta)$.

Now, we would like to define the terms in the n-th row of the modified Pascal's triangle to be $[p_{n,1}, p_{n,2}, p_{n,3}, \cdots p_{n,n+1}]$. By construction, these are equal to the terms of the corresponding row of the standard Pascal's triangle, except that the terms $[p_{n,i}]$ that have $i \equiv 0,3 (\bmod 4)$ are assigned negative signs.

Since the terms of the n-th row in the standard Pascal's triangle are $[\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots \binom{n}{n}]$, applying the transformation from the standard Pascal's triangle to the modified version, we can find that the terms of row n of the modified Pascal's triangle are $[p_{n,1}, p_{n,2}, p_{n,3}, p_{n,4}, p_{n,5}, p_{n,6}, \cdots] = [\binom{n}{0}, \binom{n}{1}, -\binom{n}{2}, -\binom{n}{3}, \binom{n}{4}, \binom{n}{5}, \cdots]$.

With this knowledge, we return to the equations we found for $cos(n\theta)$ and $sin(n\theta)$.

Substituting these $p_{n,i}$ into the earlier formulas, we get

$$\cos(n\theta) = p_{n,1}c^n + p_{n,3}c^{n-2}s^2 + p_{n,5}c^{n-4}s^4 + p_{n,7}c^{n-6}s^6 + \cdots$$
$$\sin(n\theta) = p_{n,2}c^{n-1}s^1 + p_{n,4}c^{n-3}s^3 + p_{n,6}c^{n-5}s^5 + p_{n,8}c^{n-7}s^7 + \cdots$$

These equations are the equations we wanted to prove to be true. Since we derived them from other true equations, these equations must be true as well.

Thus, the coefficients in odd positions correspond to $cos(n\theta)$, and the coefficients in even positions correspond to $sin(n\theta)$. Since these equations are derived directly from De Moivre's Theorem, they must be true for all n.

Theorem 2.

$\sum_{n=1}^{k} \cos(n\theta) = \sum_{n=1}^{k} \binom{n}{0} c^n + \sum_{n=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=2n}^{k} (-1)^n \binom{m}{2n} c^{m-2n} s^{2n}$, and

$\sum_{n=1}^{k} \sin(n\theta) = \sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=2n+1}^{k} (-1)^n \binom{m}{2n+1} c^{m-(2n+1)} s^{2n+1}$.

Proof.

Before proceeding, recall that $P_{a,b}$ denotes the b-th entry in the a-th row of the modified Pascal's triangle.

For the cosine case, let us expand the first few terms of this series for an arbitrarily large n, and we will still use the shorthand c and s notation.

$$\sum_{n=1}^{k} \cos(n\theta) = (p_{1,1}c) + (p_{2,1}c^2 + p_{2,3}s^2) + (p_{3,1}c^3 + p_{3,3}cs^2) + (p_{4,1}c^4 + p_{4,3}c^2s^2 + p_{4,5}s^4) + \cdots$$

Grouping terms according to powers of s, we obtain:

$$(p_{1,1}c + p_{2,1}c^2 + p_{3,1}c^3 + \cdots) + (p_{2,3}s^2 + p_{3,3}cs^2 + p_{4,3}c^2s^2 + \cdots) + (p_{4,5}s^4 + p_{5,5}cs^4 + p_{6,5}c^2s^4 + \cdots) + \cdots.$$

Recalling the definition for $P_{a,b}$ as a term of the modified Pascal's triangle and using the combinatoric definition of the terms of the normal Pascal's triangle, we know that $p_{a,b} = \binom{a}{b-1}$ or $-\binom{a}{b-1}$, since the terms of the modified Pascal's triangle are either the original or negated value of the corresponding term in the normal Pascal's triangle. Specifically, we know that $p_{a,b} = \binom{a}{b-1}$ if $b \equiv 1, 2 \ (\bmod 4)$ and $p_{a,b} = -\binom{a}{b-1}$ if $b \equiv 0, 3 \ (\bmod 4)$.

By the definition of the modified Pascal's triangle, each coefficient satisfies

$$p_{a,b} = \begin{cases} \binom{a}{b-1}, & b \equiv 1, 2 \ (mod\ 4), \\ -\binom{a}{b-1}, & b \equiv 0, 3 \ (mod\ 4). \end{cases}$$

Substituting these into the expansion gives:

$$\left(\binom{1}{0}c + \binom{2}{0}c^2 + \binom{3}{0}c^3 + \cdots\right) + \left(-\binom{2}{2}s^2 - \binom{3}{2}cs^2 - \binom{4}{2}c^2s^2 - \cdots\right) + \left(\binom{4}{4}s^4 + \binom{5}{4}cs^4 + \binom{6}{4}c^2s^4 + \cdots\right) + \cdots$$

This can be rewritten as

$$\sum_{n=1}^{k} \binom{n}{0} c^n + \sum_{n=2}^{k} -\binom{n}{2} c^{n-2} s^2 + \sum_{n=4}^{k} \binom{n}{4} c^{n-4} s^4 - \sum_{n=6}^{k} \binom{n}{6} c^{n-6} s^6 + \cdots$$

Note that the first summation term differs slightly in structure since the constant term corresponding to $cos(0\theta)$ is excluded.

While taking this deviation into account, we can further simplify this into:

$$\sum_{n=1}^{k} \binom{n}{0} c^n + \sum_{n=1}^{\lfloor \frac{k}{2} \rfloor} \sum_{m=2n}^{k} (-1)^n \binom{m}{2n} c^{m-2n} s^{2n}$$

And the cosine case is proven.

For the sine case, the first few terms are:

$$(p_{1,2}s) + (p_{2,2}cs) + (p_{3,2}c^2s + p_{3,4}s^3) + (p_{4,2}c^3s + p_{4,4}cs^3) + (p_{5,2}c^4s + p_{5,4}c^2s^3 + p_{5,6}s^5) \cdots$$

Regrouping by powers of s:

$$(p_{1,2}s + p_{2,2}cs + p_{3,2}c^2s + \cdots) + (p_{3,4}s^3 + p_{4,4}cs^3 + p_{5,4}c^2s^3 + \cdots) + (p_{5,6}s^5 + p_{6,6}cs^5 + p_{7,6}c^2s^5 + \cdots) + \cdots$$

Replacing $P_{a,b}$ with the corresponding binomial coefficients, we obtain:

$$\left(\binom{1}{1}s + \binom{2}{1}cs + \binom{3}{1}c^2s + \cdots\right) + \left(-\binom{3}{3}s^3 - \binom{4}{3}cs^3 - \binom{5}{3}c^2s^3 - \cdots\right) + \left(\binom{5}{5}s^5 + \binom{6}{5}cs^5 + \binom{7}{5}c^2s^5 + \cdots\right) + \cdots$$

This can be similarly simplified as:

$$\sum_{n=1}^{k} \binom{n}{1} c^{n-1} s - \sum_{n=3}^{k} \binom{n}{3} c^{n-3} s^3 + \sum_{n=5}^{k} \binom{n}{5} c^{n-5} s^5 + \cdots$$

Here, the deviation present in the sum of cosines is not present. We can simplify again into:

$$\sum_{n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{m=2n+1}^{k} (-1)^n \binom{m}{2n+1} c^{m-(2n+1)} s^{2n+1}$$

And the sine case is proven.

Theorem 3.

$$(\sin\theta + \cos\theta)^n - (\sin(n\theta) + \cos(n\theta)) = \sum_{k=0}^{\lfloor \frac{n-2}{4} \rfloor} 2\binom{n}{4k+2} c^{n-(4k+2)} s^{4k+2} + \sum_{k=0}^{\lfloor \frac{n-3}{4} \rfloor} 2\binom{n}{4k+3} c^{n-(4k+3)} s^{4k+3}$$

Proof.

From the formulas above, let the -th row of the modified Pascal's triangle be $[y_1, y_2, y_3, \ldots, y_{n+1}]$. Then, we obtain $\sin n\theta + \cos n\theta = y_1c^n + y_2c^{n-1}s + y_3c^{n-2}s + \cdots = \sum_{k=1}^{n+1} y_k c^{n-(k-1)} s^{k-1}$.

Using the definition of the modified Pascal's triangle, this series can be rewritten as
$\binom{n}{0}c^n + \binom{n}{1}c^{n-1}s - \binom{n}{2}c^{n-2}s^2 - \binom{n}{3}c^{n-3}s^3 + \cdots$.

On the other hand, from the binomial theorem, the expansion of $(\sin\theta + \cos\theta)^n$ is
$\binom{n}{0}c^n + \binom{n}{1}c^{n-1}s + \binom{n}{2}c^{n-2}s^2 + \cdots = \sum_{k=0}^{n} \binom{n}{k} c^{n-k} s^k$.

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Therefore, their difference is

$$(\sin\theta+\cos\theta)^n-(\sin n\theta+\cos n\theta)$$
$$=\left(\binom{n}{0}c^n+\binom{n}{1}c^{n-1}s+\binom{n}{2}c^{n-2}s^2+\binom{n}{3}c^{n-3}s^3+\cdots\right)$$
$$-\left(\binom{n}{0}c^n+\binom{n}{1}c^{n-1}s-\binom{n}{2}c^{n-2}s^2-\binom{n}{3}c^{n-3}s^3+\cdots\right)$$

The terms with coefficient $\binom{n}{k}$ where k=0,1 (mod 4) are the same on both sides of the subtraction (and thus cancel out), while the terms with k=2,3 (mod 4) are negated on the right side of the subtraction, leading to those terms being doubled in the result.

Using that observation, we can express the difference as $\sum_{k=0}^{\lfloor\frac{n-2}{4}\rfloor}2\binom{n}{4k+2}c^{n-(4k+2)}s^{4k+2}+\sum_{k=0}^{\lfloor\frac{n-3}{4}\rfloor}2\binom{n}{4k+3}c^{n-(4k+3)}s^{4k+3}$ and the theorem is true.

Examples

The fifth row of the modified Pascal's Triangle is $\cos^5(\theta)$, $5\cos^4(\theta)\sin(\theta)$, $-10\cos^3(\theta)\sin^2(\theta)$, $-10\cos^2(\theta)\sin^3(\theta)$, $5\cos(\theta)\sin^4(\theta)$, $\sin^5(\theta)$.

From this, we can use the formulas we proved to find $\cos(5\theta)$ and $\sin(5\theta)$.

$\cos(5\theta) = \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta)$ taking terms 1, 3, and 5, and

$\sin(5\theta) = 5\cos^4(\theta)\sin(\theta) - 10\cos^2(\theta)\sin^3(\theta) + \sin^5(\theta)$, taking terms 2, 4, and 6.

For examples of the formulas for $\sum_{n=1}^{k}\cos(n\theta)$ and $\sum_{n=1}^{k}\sin(n\theta)$ we calculate with k=3 :

$$\sum_{n=1}^{3}\cos(n\theta)=\sum_{n=1}^{3}\binom{n}{0}c^n+\sum_{n=1}^{\lfloor\frac{3}{2}\rfloor}\sum_{m=2n}^{3}(-1)^n\binom{m}{2n}c^{m-2n}s^{2n}$$
$$=\sum_{n=1}^{3}\binom{n}{0}c^n+\sum_{m=2}^{3}(-1)\binom{m}{2}c^{m-2}s^2$$
$$=c+c^2+c^3-\binom{2}{2}c^{2-2}s^2-\binom{3}{2}c^{3-2}s^2$$
$$=c+c^2+c^3-s^2-3cs^2$$
$$=c+(c^2-s^2)+(c^3-3cs^2)$$
$$=\cos(\theta)+(\cos^2(\theta)-\sin^2(\theta))+(\cos^3(\theta)-3\cos(\theta)\sin^2(\theta))$$
$$=\cos(\theta)+\cos(2\theta)+\cos(3\theta),$$

$$\sum_{n=1}^{3}\sin(n\theta)=\sum_{n=0}^{\lfloor\frac{3-1}{2}\rfloor}\sum_{m=2n+1}^{3}(-1)^n\binom{m}{2n+1}c^{m-(2n+1)}s^{2n+1}$$
$$=\sum_{m=1}^{3}(-1)^0\binom{m}{1}c^{m-1}s^1+\sum_{m=3}^{3}(-1)^1\binom{m}{3}c^{m-3}s^3$$
$$=s+2cs+3c^2s-s^3$$
$$=\sin(\theta)+(2\sin(\theta)\cos(\theta))+(3\cos^2(\theta)\sin(\theta)-\sin^3(\theta))$$
$$=\sin(\theta)+\sin(2\theta)+\sin(3\theta).$$

For an example of the proven formula for the difference between $(\cos\theta+\sin\theta)^n$ and $\cos(n\theta)+\sin(n\theta)$, we can calculate the case where n=5 :

$$\sum_{k=0}^{\lfloor\frac{5-2}{4}\rfloor}2\binom{5}{4k+2}c^{5-(4k+2)}s^{4k+2}+\sum_{k=0}^{\lfloor\frac{5-3}{4}\rfloor}2\binom{5}{4k+3}c^{5-(4k+3)}s^{4k+3}$$
$$=\sum_{k=0}^{0}2\binom{5}{4k+2}c^{5-(4k+2)}s^{4k+2}+\sum_{k=0}^{0}2\binom{5}{4k+3}c^{5-(4k+3)}s^{4k+3}$$
$$=2\binom{5}{2}c^{5-(2)}s^2+2\binom{5}{3}c^{5-(3)}s^3$$
$$=20c^3s^2+20c^2s^3$$
$$=20\cos^3\theta\sin^2\theta+20\cos^2\theta\sin^3\theta$$

Which is indeed true if we manually calculate $(\cos\theta+\sin\theta)^5$-$(\cos(5\theta)+\sin(5\theta))$.

Conclusion

In this paper, we showed how a modified version of Pascal's Triangle can reveal new insights into classical trigonometric identities. By altering the signs of specific entries in each row, we demonstrated that the odd and even coefficients naturally correspond to the expansions of $\cos(n\theta)$ and $\sin(n\theta)$. These results were established in two complementary ways: through induction using the addition formulas for sine and cosine, and through an alternative proof based on De Moivre's Theorem.

Although these identities are familiar to experts, the modified Pascal's Triangle establishes a straightforward combinatorial framework that not only clarifies the relationships but also reorganizes advanced trigonometric expansions into an accessible form for middle and high school students.

In particular, we showed that the expansions of $\cos(n\theta)$ and $\sin(n\theta)$ can be fully expressed in a purely combinatorial way using the modified Pascal's Triangle. Moreover, by comparing the expansions of $(\cos\theta+\sin\theta)^n$ through the standard Pascal's Triangle with those of $\cos(n\theta)+\sin(n\theta)$ through the modified Pascal's Triangle, we established a direct connection between the two expressions. This produces the classical results; however, it also highlights the value of presenting trigonometry through a combinatorial lens.

For further research, this approach could be extended to investigate other trigonometric functions, such as tangent or secant, or even generalized to hyperbolic functions. In addition, the modified Pascal's Triangle may serve as a starting point for exploring deeper combinatorial links with well-studied structures like Chebyshev polynomials, Fibonacci numbers, or other polynomial families.[4,5] Such directions would not only broaden the scope of this work but also strengthen the bridge between combinatorics and trigonometry in creative and accessible ways.

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Authors

Sihwan Lee, a 10th-grade high school student, has a profound passion for pure mathematics. He is deeply engaged in studying trigonometry and discrete mathematics, excelling particularly in probability and combinatorics. Sihwan also has a strong interest in physics, with a love for dynamics and thermodynamics, and aspires to become an engineer in the future.