

An Intuitive and General Learning Approach for Function Transformations

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ABSTRACT: This paper proposes an intuitive yet broadly applicable learning framework for understanding function transformations, a mathematical topic that is widely misunderstood by high school students. Traditional approaches often rely on memorization-based rules that conflict with students' intuition and lead to frequent application errors and weak long-term retention. Drawing from documented shortcomings of such rote techniques, this research introduces a simple, logical, and conceptually grounded approach centered on point mapping. Rather than presenting transformation rules as isolated formulas, the framework justifies the effects of transformations by directly examining how individual points on a function are affected. This perspective encourages learners to reason through transformations systematically, aligning algebraic manipulation with geometric interpretation. By focusing on the relationship between input values and corresponding outputs, the approach helps reconcile the counterintuitive nature of horizontal shifts, stretches, and reflections. The proposed procedures are supported by graphical representations that visually reinforce the consistency and validity of the derived principles across arbitrary functions. Overall, the framework promotes a clear, transferable thought process for applying function transformations in problem-solving contexts, while fostering deeper comprehension and more durable learning outcomes for higher-level secondary mathematics learning.

KEYWORDS: Mathematics, Algebra, Function Transformations, Horizontal Translation and Scaling, Point Mapping.

■ Introduction

Function transformation refers to a systematic modification of a given function that produces a new function by altering its input, output, or both. These transformations correspond to predictable changes in the graph of the function, such as translations, scaling, reflections, or combinations thereof, while preserving the underlying relationship between variables. From the point mapping perspective, a point (x, y) in a two-dimensional space is mapped to a new point $(\tilde{x}, \tilde{y})$ through function transformations (Figure 1).

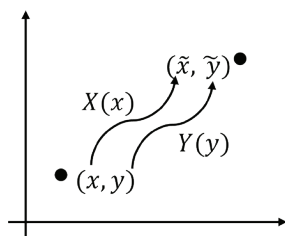


Figure 1: Function transformations map from original points to new points by a mapping function pair $(X(\cdot), Y(\cdot))$. The figure demonstrates how the transformation induces a deterministic coordinate remapping that is applied to each axis. This highlights the resulting geometric deformation and its effect on relative spatial positions.

Function transformations are an essential topic for high school students because they provide conceptual insight, analytical tools, and transferable skills that support both advanced mathematics and practical applications. This topic helps students understand functions as relationships between inputs and outputs, develop algebraic reasoning and generalization skills, and recognize the important connection between symbolic

expressions and graphical representations. It also strengthens students' ability to model and solve real-world problems. However, function transformations remain a persistent challenge for students in secondary mathematics, particularly in Algebra II and pre-calculus. Despite procedural fluency, many students struggle to understand and correctly apply transformation rules, especially for horizontal shifts, stretches, and reflections. These difficulties have been widely documented and are often attributed to the counterintuitive nature of transformations and the way they are commonly taught.^{1,2}

A major source of this problem lies in typical instructions in most high schools that emphasize memorization over reasoning.³ Transformation rules are frequently presented as formulas to recall rather than as results of algebraic relationships between inputs and outputs.^{4,5} As a result, students tend to rely on fragile heuristics that often fail in unfamiliar contexts, leading to misconceptions and weak retention. While prior research has proposed the use of such visual strategies as graph comparisons and dynamic technology to build intuition, these approaches often prioritize global pattern recognition over the explicit articulation of the algebraic mechanisms underlying transformations.⁶⁻⁹ In contrast, this paper builds directly on that body of work by retaining the value of multiple representations while making the point-by-point mapping between pre-images and images explicit, thereby addressing a gap left implicit in these studies.

Research on function understanding further emphasizes that students should conceptualize functions as relationships rather than static graphs.¹⁰⁻¹² While these studies successfully broaden students' perspectives on functions, they often stop short of

formalizing how individual points are transformed under algebraic operations. Similarly, scholars who critique surface-level instruction argue that neither symbolic manipulation nor graphical recognition alone is sufficient for meaningful learning without grounding in underlying reasoning.¹³⁻¹⁵ This paper extends these arguments by proposing a systematic way to connect symbolic, graphical, and relational perspectives through a single unifying principle: pointwise reasoning about how inputs are mapped to outputs under transformation.

Noticeably, the learning gap on this topic is particularly evident in horizontal transformations and in distinguishing between input scaling (horizontal effects) and output scaling (vertical effects). Existing studies have identified these misconceptions but have not provided a fully generalizable framework that resolves them across representations. In response, this paper introduces a unified framework for understanding function transformations through pointwise reasoning. Unlike approaches that emphasize pattern recognition or rule memorization, this framework models transformations explicitly as mappings between domain and range values, thereby providing an algebraic justification for standard transformation rules. At the same time, it aligns with prior research by integrating graphical intuition and supporting the use of multiple representations, while extending earlier work by making the underlying logic transparent and transferable. Because the framework applies to arbitrary functions and maintains consistency across symbolic, graphical, and contextual representations, it offers a coherent alternative to fragmented instructional approaches. In doing so, this framework aims to improve conceptual understanding, accuracy, and transferability in students' learning of function transformations.

■ Methods

We define (x, y) as a point on the curve defined by the function $f(x)$ and $(\tilde{x}, \tilde{y})$ as a point on the new curve $g(\tilde{x})$, such that:

$$\begin{cases} y = f(x) \\ \tilde{y} = g(\tilde{x}) \end{cases}$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are two functions governing (x, y) and $(\tilde{x}, \tilde{y})$, respectively.

In Algebra II, the general relationship between $f(\cdot)$ and $g(\cdot)$ can be characterized as: $g(x) = af(bx+c)+d$. Note that (1) the parameter a is defined as a vertical scaling or dilation in the direction of the y-axis; (2) d is a vertical shift or translation; (3) $b \neq 0$ is a horizontal scaling or dilation in the direction of the x-axis, and (4) c is a horizontal translation. Therefore, the function transformation question of interest becomes how to find the mappings in the x- and y-coordinates, given:

$$\begin{cases} y = f(x) \\ \tilde{y} = af(b\tilde{x}+c)+d \end{cases} \quad (1)$$

where (x, y) and $(\tilde{x}, \tilde{y})$ are the coordinates of the original and transformed points, respectively.

Vertical Transformation:

When only the vertical transformation is considered in Algebra II, it is simply represented as:

$$g(x) = af(x) + d$$

Given $y = f(x)$, and $\tilde{y} = g(x) = af(x) + d$, it implies:

$$\tilde{y} = Y(y) = ay + d \quad (2)$$

Note the y-coordinate of the original point is first scaled by a and then shifted in the y-direction by d . The sequence of transformations follows the standard operator precedence, with multiplication applied before addition. This form implies that for $0 < a < 1$, the transformation scales the function vertically by a factor less than one, producing a compression in the y-direction. When $a > 1$, the function is stretched out in the y-direction. Different choices of parameter values lead to noticeably different vertical behaviors in the transformed function (Table 1). Specifically, the function is vertically compressed when $|a| < 1$. For example, when $a = 0.5$, the graph is compressed by half in the y-direction. Similarly, the function is stretched vertically when $|a| > 1$. An example demonstrating this point is that $a = 2$ results in a graph dilated by 2 in the y-direction. In particular, the function is vertically reflected along the x-axis when $a < 0$. Furthermore, the parameter d produces a vertical translation, shifting the function upward for $d > 0$ and downward for $d < 0$.

Overall, the relatively intuitive nature of vertical transformations makes it easier to understand than the horizontal transformations, which are more often the source of confusion.

Table 1: Given that vertical scaling is performed prior to vertical shifting in the transformation sequence, changes in the parameters a and d create distinct vertical effects on the function graph. All parameter combinations of a and d are enumerated in terms of their resultant reflection in the y-direction, vertical scaling effects, and vertical translation behaviors, providing a clear correspondence between specific parameter values and the resulting transformation outcomes.

(1) Parameter a	Reflection in Y-direction	Scale in Y-direction	(2) Parameter d		Shift in Y-direction
$a < -1$	Yes	Stretch	$d > 0$	$d < 0$	Up
$a = -1$		Neither			
$-1 < a < 0$		Compress			
$0 < a < 1$	No	Compress	$d > 0$	$d < 0$	Down
$a = 1$		Neither			
$a > 1$		Stretch			

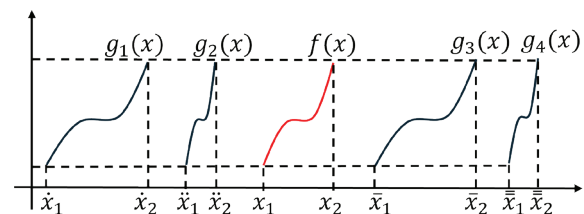


Figure 2: A geometric comparison of the original function $f(x)$ (red) with its horizontally scaled and translated counterparts $g_i(x), i=1,2,3,4$ (black), including both compression and stretch cases for generality. The figure illustrates how horizontal scaling modifies the input domain by a reciprocal factor, resulting in visibly distinct rates of expansion or contraction across the transformed graphs. These comparisons highlight the predictable displacement of key geometric features under horizontal translation and clarify how scaling and shifting jointly structure the horizontal deformation behavior.

Horizontal Transformation:

Without loss of generality, the original and horizontally transformed functions are denoted as $f(x)$, and $g(x) = f(bx+c)$, respectively. Additionally, the x-coordinate of the original point and that of the mapped point after the horizontal transformation are x and $\tilde{x}$, respectively. This paper presents a learning approach to function transformation that emphasizes the conceptual core of horizontal transformation by interpreting it as the process of identifying a value $\tilde{x}$ such that $f(x)$ and $g(\tilde{x})$

yield the same output. In other words, this key relationship can be simply represented by the following equation:

$$f(x) = g(\tilde{x}) \quad (3)$$

It is this equation that suggests the start of an intuitive but generic way to understand the horizontal transformation. Although Equation (3) seems to be obvious to a certain degree, it is not well understood and therefore not highlighted in the function transformation learning process by the vast majority of learners. This algebraic relationship is illustrated in Figure 2, where the original function $f(x)$ is shifted and scaled horizontally in different ways. The original function $f(x)$ has two endpoints with the x-component being x_1 and x_2 , respectively. The horizontally transformed functions $g_1(x), \dots, g_4(x)$ are horizontally (1) stretched and shifted to the left; (2) compressed and shifted to the left; (3) stretched and shifted to the right; and (4) compressed and shifted to the right, respectively.

A common source of confusion among learners regarding horizontal transformations is determining whether horizontal scaling or horizontal translation should be applied first. This ambiguity often arises because the order of operations for horizontal transformations is not as intuitively understood as their vertical counterparts. The proposed learning framework resolves this issue by deriving the mapping relationship through a systematic algebraic approach. By applying the inverse function $f^{-1}(\cdot)$ to both sides of Equation (3), we show how an input value is transformed:

$$f^{-1}(f(x)) = f^{-1}(g(\tilde{x})) \quad (4)$$

As $g(x) = f(bx+c)$, it suggests:

$$x = f^{-1}(f(x)) = f^{-1}(g(\tilde{x})) = f^{-1}(f(b\tilde{x}+c)) = b\tilde{x}+c$$

Table 2: Different values of b and c yield different horizontal transformation effects in the execution order of the horizontal translation followed by the horizontal scaling. All parameter combinations of b and c are enumerated in terms of their resultant horizontal shift, reflection and scaling effects in the x-direction, establishing a clear linkage between each parameter choice and the corresponding transformation behavior.

(1) Parameter c	Shift in X-direction	(2) Parameter b	Reflection in X-direction	Scale in X-direction
$c < 0$	Right	$b < -1$	Yes	Compress
		$b = -1$		Neither
		$-1 < b < 0$		Stretch
		$0 < b < 1$		Stretch
$c = 0$	Neither	$b = 1$	No	Neither
		$b > 1$		Compress
		$b < -1$		Compress
		$b = -1$		Neither
$c > 0$	Left	$-1 < b < 0$	Yes	Stretch
		$0 < b < 1$		Stretch
		$b = 1$		Neither
		$b > 1$		Compress
		$b < -1$	Yes	Compress
		$b = -1$		Neither
		$-1 < b < 0$		Stretch
		$0 < b < 1$		Stretch
		$b = 1$	No	Neither
		$b > 1$		Compress
		$b < -1$		Compress
		$b = -1$		Neither

Hence:

$$\tilde{x} = \frac{1}{b}(x - c) \quad (5)$$

By Equations (2) and (5), the mapping between (x, y) and $(\tilde{x}, \tilde{y})$ subject to the System (1) can be represented by:

$$(x, y) \Rightarrow (X(x) = \tilde{x} = \frac{1}{b}(x - c), Y(y) = \tilde{y} = ay + d) \quad (6)$$

The horizontal mapping results from Equation (6) clearly suggest that the problem-solver must first conduct a horizontal translation by the absolute value of c in the opposite direc-

tion of its original sign (a negative sign indicates a translation right, while a positive sign indicates a translation left) before scaling the x-value by a factor of $\frac{1}{b}$, abiding by order of operation rules. This helps clarify misconceptions about what values are involved in manipulating input and output values to achieve the final image and the order in which they are applied. While the sign of the parameter c suggests the translation direction in the first step, the parameter b controls the scaling effect that follows. When $\frac{1}{|b|} < 1$, the function is horizontally compressed due to $1/|b| < 1$. Likewise, the function is next stretched in the x-direction when $0 < |b| < 1$. Additionally, the horizontal transformation is reduced to horizontal translation when $b=1$. Similarly, when $c=0$, the horizontal transformation becomes equivalent to applying only the horizontal scaling. By the transformation execution order, the different horizontal transformation effects by different values of b and c are summarized in Table 2.

■ Results

Validation of Proposed Horizontal Transformation Framework:

This paper primarily discusses the mapping relationships associated with horizontal function transformations, which can be summarized as follows.

Question 1: Given $f(x)$, and $g(x) = f(bx+c)$, find the mappings of points on curves by these two functions.

To validate the derived results, we visually check the results by taking the following function $f(x) = \tanh(x)$ and $g(x) = f(8x+0.5)$ in Figure 3. By Equation (6), we have such mappings: $(x, y) \Rightarrow (X(x) = \tilde{x} = \frac{1}{b}(x - c) = 1/8(x - 0.5), Y(y) = \tilde{y} = ay + d = (1)y + (0) = y)$. Four randomly sampled but representative point pairs, (A, a), (B, b), (C, c), and (D, d), corresponding to $x = -2, -1, 0.6$, and 1 , are used to validate the resulting mappings. These pairs are superimposed on the graphs of $f(x)$ and $g(x)$ to enable visual verification. The validity of the mapping relationships is manifested by observing that points of A, B, C, and D in the domain correspond consistently and precisely to their intended locations at a, b, c, and d in the codomain. This one-to-one and well-defined correspondence ensures that no elements are misplaced, duplicated, or lost in the transformation. In this way, the correctness of the mapping relationships is verified. Implementation of the dynamic horizontal transformation process for $g(x) = f(bx+c)$ in a point-by-point manner is created in Desmos (<https://www.desmos.com/calculator/kkl5dlxijz>).

A Frequently Confusing Companion Question on Horizontal Transformations:

In contrast to Question 1, a commonly misunderstood companion horizontal transformation problem is presented below.

Question 2: Given $f(x)$ and $g(x) = f(b(x+c))$, find the mappings of points on curves by these two functions.

Note the difference between the expression of the input to the function $f(x)$ in this question and that in Question 1. Using the proposed framework, we show that this problem can be efficiently resolved, with two solution methods provided for comparative validation.

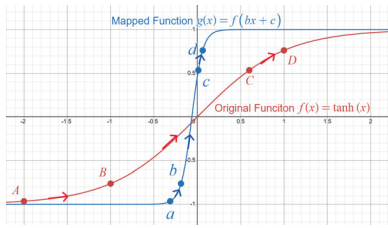


Figure 3: Given $f(x)=\tanh(x)$ (red) and $g(x)=f(8x+0.5)$ (blue), the resulting mappings are validated by overlaying onto the graphs of $f(x)$ and $g(x)$ four pairs of points, i.e. (A, a), (B, b), (C, c), and (D, d), corresponding to $x=-2, -1, 0.6$, and 1 , respectively. The verified preservation of domain–codomain structure under the prescribed mapping confirms that Equation (6) provides an accurate algebraic representation of the underlying geometric transformation.

Solution 1: The given problem can be converted to the problem setup in the Method section by factoring and expressing the argument of the function:

$$g(\tilde{x}) = f(b(\tilde{x} + c)) = f(b\tilde{x} + bc)$$

Note that the parameter b in this question is the parameter b in the Method section, while the parameter c in the Method section becomes bc in this question. By Equation (5), we have the following conclusion:

$$\tilde{x} = \frac{1}{b}(x - bc) = \frac{1}{b}x - c$$

This solution indicates that the x -coordinate of the original point is first mapped to $\tilde{x}$ through a horizontal translation by bc , followed by a scaling factor of $\frac{1}{b}$. Equivalently, the same transformation can be achieved by first scaling x by $\frac{1}{b}$ and then applying a horizontal shift by c .

Solution 2: By Equation (3), the essence of the relationship between $f(x)$ and $g(\tilde{x})$ can be represented as:

$$f(x) = g(\tilde{x}) = f(b(\tilde{x} + c))$$

Taking f^{-1} on both sides of the equation above, we have $x = b(\tilde{x} + c)$. Hence, $\tilde{x} = \frac{1}{b}x - c = \frac{1}{b}(x - bc)$. This solution is therefore equivalent to Solution 1.

In Figure 4, we visualize the results by taking the following function $f(x)=\tanh(x)$ and $g(x)=f(8(x+0.5))$ with solution mappings $(x, y) \Rightarrow (x(x) = \tilde{x} = \frac{1}{b}(x - bc) = \frac{1}{8}(x - (8)(0.5)) = \frac{1}{8}(x - 4), Y(y) = \tilde{y} = ay + d = (1)y + (0) = y)$. For validation of the resulting mappings, four random pairs of points $P=(x, y)$, and $Q=(\tilde{x}, \tilde{y})$ associated with $x=-0.3, 0, 0.5$, and 2 are overlaid with the plots of $f(x)$ and $g(x)$. The validity of the mapping relationships is thereby established through the consistent alignment of all points with their intended locations. The dynamic process of the horizontal transformation $g(x)=f(b(x+c))$ is implemented in a point-by-point perspective on Desmos (<https://www.desmos.com/calculator/okbx1kyfms>).

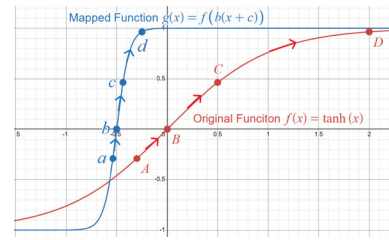


Figure 4: Given function $f(x)=\tanh(x)$ (red) and $g(x)=f(8(x+0.5))$ (blue), the resulting mappings are validated by overlaying onto the graphs of $f(x)$ and $g(x)$ four pairs of points, i.e. (A, a), (B, b), (C, c), and (D, d), corresponding to $x=-0.3, 0, 0.5$, and 2 , respectively. The confirmed preservation of the domain–codomain structure supports the efficacy of the proposed comprehension framework derived from Equation (3).

A Full Example of Function Transformation Using the Proposed Learning Approach:

A complete illustrative plot of the letter “K,” constructed from three function segments and involving both horizontal and vertical transformations, is presented next. Accordingly, the following functions are used to represent the three strokes of the letter:

$$u_1(x) = 10x, 0 \leq x \leq 0.1; \quad u_2(x) = 1.2x + 0.45, 0.05 \leq x \leq 0.45; \quad u_3(x) = -1.5x + 0.5, 0.05 \leq x \leq 0.3$$

For each segment, two endpoints are specified. For instance, points A and B correspond to $u_1(x)$, points C and D to $u_2(x)$, and points E and F to $u_3(x)$. The following transformation is then applied to each function:

$$\tilde{u}_i(x) = au_i(bx + c) + d, \quad i = 1, 2, 3 \quad (7)$$

Specifically, different combinations of the parameter values are employed, as summarized in Table 3. Under parameter set (1), all strokes undergo vertical compression, horizontal stretching, and a translation toward the lower right. Using parameter set (2), the strokes are vertically stretched, horizontally compressed, and translated toward the upper left. Finally, parameter set (3) results in a horizontal reflection followed by a translation toward the lower left. The resulting variations of the letter “K” produced by these transformations are illustrated in Figure 5.

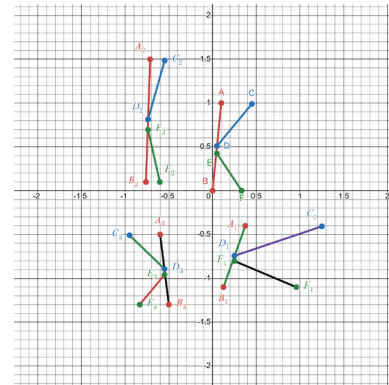


Figure 5: Different versions of letter “K” after horizontal and vertical transformations are presented along with the end points for each stroke, where $A_i, B_i, \dots, F_i$ are transformed end points with the parameter set i . Equation (7) governs the stroke-level transformation, while Equation (6) determines the mapping of the corresponding endpoints. The precise alignment of all mapped points with their intended target positions confirms the correctness of the mapping relations and supports the effectiveness of the proposed learning approach. This agreement between theoretical predictions and observed outcomes reinforces the reliability of the proposed transformation framework as a tool for developing conceptual understanding.

Table 3: Combinations of parameter values used for transformed letter “K” in Figure 5 are presented. The validated set of ‘K’ renderings induced by horizontal and vertical transformations corroborates the efficacy of the proposed learning approach for conceptualizing function mappings. The preservation of predicted geometric features under axis-specific operations provides evidence of rule-consistent reasoning rather than rote pattern matching.

Parameter Set	Parameter a	Parameter b	Parameter c	Parameter d
(1)	0.7	0.4	-0.05	-1.1
(2)	1.4	2.1	1.6	0.1
(3)	0.8	-1	-0.5	-1.3

While the letter strokes are transformed according to Equation (7), all corresponding endpoints are mapped using Equation (6). The correctness of the mapping relationships is therefore validated by the observation that all points are mapped to their intended locations. The dynamic process of the horizontal transformation $g(x)=f(b(x+c))$ is implemented in a point-by-point mapping perspective on Desmos (<https://www.desmos.com/calculator/xidqjp5evb>)

The Net Vertical Relationship between the Original and Transformed Points:

This paper presents the vertical transformation from a point-to-point mapping perspective in the Method section. While Table 1 summarizes vertical transformation effects due to different choices of parameter values for a and d by the execution order (i.e., scaling first and shifting next), the actual vertical relationship between the original and mapped points is worth further analysis. Specifically, it is noted that the resulting vertically transformed point may be either above or below the original point according to a different set of rules.

To find such rules, we can simply solve:

$$\tilde{y} = ay + d \gtrless y$$

Combining the similar terms, we further have:

$$(1-a)y \gtrless d$$

Therefore, the rules that determine the relative vertical position between the vertically transformed point and the original point can be readily derived and are summarized in Table 4. It is noticed that such a vertical relationship is not only determined by the parameter a and d , but also the y -coordinate of the original point given $a \neq 1$. When $a=1$, the vertical relationship between y and $\tilde{y}$ only depends on the sign of the parameter d .

Table 4: The net vertical relationship between the vertically transformed point and the original point is summarized. This table makes it explicit how vertical scaling and vertical shifting jointly determine the final position of each mapped point, thereby clarifying the combined effect of the parameters on the transformation outcome.

Parameter a	Y-coordinate of the original point	Net Vertical Relationship
$a < 1$	$y \gtrless \frac{d}{1-a}$	$\tilde{y} \gtrless y$
$a = 1$	$d \gtrless 0$	$\tilde{y} \gtrless y$
$a > 1$	$y \gtrless \frac{d}{1-a}$	$\tilde{y} \gtrless y$

Table 5: The net horizontal relationship between the horizontally transformed and the original point is summarized. This table delineates how the interaction between horizontal scaling and horizontal shifting governs the final coordinates of each mapped point, providing a transparent depiction of the joint parameter effects on the transformation.

Parameter b	X-coordinate of the original point	Net Horizontal Relationship
$b < 0$ or $b > 1$	$x \gtrless \frac{c}{1-b}$	$\tilde{x} \gtrless x$
$0 < b < 1$	$x \gtrless \frac{c}{1-b}$	$\tilde{x} \gtrless x$
$b = 1$	$c \gtrless 0$	$\tilde{x} \gtrless x$

The Net Horizontal Relationship between the Original and Transformed Points:

The proposed learning framework clarifies the widespread confusion regarding the order of horizontal translation and scaling by establishing the mapping relationship through a structured algebraic derivation. This process clearly demonstrates the interaction between the translation and scaling parameters, thereby illuminating the correct operational sequence. Considering only the horizontal component of the transformation, the execution order (horizontal translation followed by horizontal scaling) produces different horizontal effects for different choices of parameter b and c . These effects are systematically summarized in Table 2. While Table 2 presents the horizontal transformation effects in the order of translation followed by scaling, it is important to note that the resulting horizontally transformed point may lie either to the left or to the right of the original point. It is noticed that $\tilde{x} = \frac{1}{b}(x - c) = \frac{x}{b} - \frac{c}{b}$, suggesting that the direction of the horizontal mapping, with respect to the original point, is governed by the values of the parameter b , c , and the x -coordinate jointly.

The critical conditions can be simply found by solving:

$$\tilde{x} = \frac{x}{b} - \frac{c}{b} \gtrless x$$

Combining the similar terms about x , one can further have:

$$\frac{1-b}{b}x \gtrless \frac{c}{b}$$

Therefore, the rules determining the relative horizontal position between the horizontally transformed point and the original point can be systematically derived and are summarized in Table 5. It is noticed that such a horizontal relationship is not only determined by the parameter b and c , but also the x -coordinate of the original point, given $b \neq 1$. When $b=1$, the horizontal relationship between x and $\tilde{x}$ only depends on the sign of the parameter c .

Application to Functions Without Inverses:

The proposed learning framework addresses the function transformation problem by deriving the mapping relationship through a structured and transparent algebraic procedure. By applying the inverse function to both sides of Equation (3), the framework uncovers the sequential mechanism through which each input value is transformed. This derivation clearly illustrates the interaction between the parameters responsible for horizontal translation and horizontal scaling, thereby making the proper execution order explicit. Through this approach, it becomes evident that horizontal translation is applied prior to horizontal scaling, eliminating confusion and offering a math-

ematically grounded understanding of horizontal function transformations.

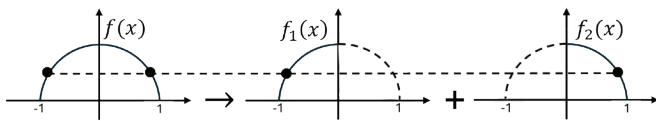


Figure 6: While the function $f(x)=\sqrt{1-x^2}$ does not have the inverse function, it can be decomposed to two functions with $x\in[-1,0)$ and $x\in[0,1]$, respectively. Since each transformation function admits an inverse, the proposed learning approach can be systematically extended to functions that do not possess an explicit inverse representation. The ability to reconstruct input-output relationships without requiring symbolic inversion highlights the flexibility and conceptual strength of the method.

Suggested by Equation (4), the proposed framework to understand the function transformation depends on the existence of the inverse function f^{-1} . If f^{-1} does not exist, the original function $f(x)$ can always be partitioned into pieces $f_i(x), i=1,2,\dots$ such that the inverse functions of all pieces $f_i^{-1}(x)$ exist. For example, the function $f(x)=\sqrt{1-x^2}$ can be re-written as:

$$f(x) = \begin{cases} f_1(x) = \sqrt{1-x^2}, & -1 \leq x < 0 \\ f_2(x) = \sqrt{1-x^2}, & 0 \leq x \leq 1 \end{cases}$$

In this way, inverse functions of both pieces exist, i.e. $f_1^{-1}(x)=\sqrt{1-x^2}$, and $f_2^{-1}(x)=\sqrt{1-x^2}$. Hence, the proposed framework for function transformation can be applied to each piece function, individually (Figure 6).

Generalization of the Proposed Learning Method:

Generalization 1: While this paper primarily discusses the common horizontal transformation formats in Algebra II (i.e., $g(x)=f(bx+c)$ or $g(x)=f(b(x+c))$), the proposed method can be applied to more general horizontal transformations. Denoting the horizontal transformation by the function $h(\cdot)$, we have:

$$y = f(x) = f(h(\tilde{x})) = \tilde{y}$$

Taking the inverse function $f^{-1}(\cdot)$ on both sides:

$$f^{-1}(f(x)) = f^{-1}(f(h(\tilde{x}))) \Rightarrow x = h(\tilde{x})$$

Taking the inverse function $h^{-1}(\cdot)$ on both sides:

$$\tilde{x} = h^{-1}(x)$$

Hence, we derive the mappings for a case more general than Equation (6):

$$(x, y) \Rightarrow (X(x) = \tilde{x} = h^{-1}(x), Y(y) = \tilde{y} = ay + d) \quad (8)$$

Generalization 2: Built on top of Equation (7), a more general mapping can be derived in which the horizontal transformation is generalized by a function $h(\cdot)$, while the vertical transformation is simultaneously generalized by a function $g(\cdot)$. Given $y=f(x)$, and $\tilde{y}=g(f(h(x)))$, the following point-to-point mapping is established:

$$(x, y) \Rightarrow (X(x) = \tilde{x} = h^{-1}(x), Y(y) = \tilde{y} = g(y)) \quad (9)$$

Discussion

While this paper outlines an intuitive and broadly applicable approach to learning function transformations, it is equally important to consider how this framework might be implemented in an actual classroom setting. A more explicit discussion of pedagogical applications would help bridge the gap between theoretical development and instructional practice, thereby making the proposed ideas more immediately useful and actionable for educators and learners alike. Although the framework is shown to apply even to functions without inverses by a piecewise decomposition strategy, this aspect of the approach warrants further reflection from an educational perspective. Specifically, the process of decomposing a function into invertible pieces is not merely a technical workaround, but a potentially rich learning opportunity in its own. Engaging students in identifying, constructing, and analyzing such piecewise decompositions can deepen their conceptual understanding of functions, domains, and transformations. Therefore, a deeper discussion of these elements helps strengthen the coherence and application validity of the overall approach.

Framework Implementation:

To enhance the practical value of this proposed framework, it is important to consider how it can be implemented in a classroom setting. One effective approach is to structure it as a guided discovery lesson that gradually leads students from intuitive observations to formalized mapping relationships. For example, a teacher might begin with a brief introductory discussion reviewing familiar transformations such as shifts, stretches, and reflections. Students can then work in small groups to identify corresponding points and record ordered pairs, noting how selected points are mapped between the two functions.

Next, the teacher may provide a structured handout or worksheet that guides students in generalizing these observations into mapping rules, encouraging them to express patterns both algebraically and visually. During this phase, the framework acts as a scaffold, helping students connect pointwise correspondences with global function transformations. To consolidate these insights, the teacher can present the proposed learning framework through slides, explicitly clarifying the reasoning process underlying the derivation of the mapping formula.

Finally, the lesson can incorporate interactive graphing activities using technology (e.g., Desmos), allowing students to dynamically test and validate their mappings. This instructional sequence, comprising introduction, guided exploration, structured generalization, and visual verification, supports students in developing a clear understanding of the formula derivation process while effectively bridging theory and practice. As a result, the framework becomes more accessible and immediately applicable in the classroom.

Reinforcement of the Piecewise Function Strategy:

This paper introduces a piecewise function strategy to apply the proposed transformation framework to irreversible functions. Importantly, this approach also creates meaningful

learning opportunities by prompting students to explore function structure more deeply and to develop a stronger conceptual understanding of the reasoning underlying transformations. Rather than viewing piecewise definitions as merely technical or cumbersome, students can be encouraged to see them as a way to break down complex relationships into manageable components. By analyzing how different intervals of the domain correspond to different transformation rules, learners develop a more nuanced understanding of how and why a single global mapping may not suffice in all cases.

In a classroom setting, teachers can leverage this by prompting students to first identify where a single transformation rule fails and then collaboratively construct separate rules for distinct regions. This process promotes mathematical sense-making, as students must justify why the function behaves differently across intervals and how each piece contributes to the overall mapping. It also reinforces the idea that functions are not always uniform but can exhibit varied behavior depending on the domain.

Additionally, working with piecewise mapping encourages students to practice precision in notation and reasoning, as they must clearly define interval boundaries and ensure continuity or intentional discontinuity where appropriate. Teachers can further extend this as a discussion point, asking students to compare piecewise and non-piecewise representations and reflect on the advantages and limitations of each.

Ultimately, incorporating the piecewise strategy transforms a potentially procedural step into an opportunity for conceptual development, fostering flexibility, analytical thinking, and a deeper appreciation of how complex transformations can be systematically understood and constructed. Therefore, we propose that the piecewise strategy be viewed not only as a methodological tool, but also as a pedagogical resource. By incorporating it into instruction, teachers can encourage students to think more flexibly about functions, recognize structural patterns, and develop stronger analytical skills. In this sense, the proposed framework does more than extend applicability. It opens the door to meaningful exploratory learning experiences that can strengthen students' mathematical reasoning and adaptability.

■ Conclusion

This work addresses a persistent gap between high school students' procedural familiarity with basic function-transformation rules and their deeper, conceptual understanding of why these rules operate as they do. Through the introduction of a fully general mathematical framework for analyzing function transformations, the study offers a coherent and logically grounded method for resolving common sources of confusion—particularly the counterintuitive behavior of horizontal transformations. This framework enables learners not only to apply transformation rules correctly but also to derive these rules from first principles, thereby reducing reliance on rote memorization and supporting the development of meaningful, long-lasting mathematical understanding.

To reinforce this conceptual approach, the study presents carefully constructed graphing examples that integrate both horizontal transformations—the primary focus of the investigation—and their vertical counterparts. These examples align directly with a pair of parallel algebraic proofs that mirror one another in structure and logic, illustrating the convergence of graphical and algebraic perspectives. Together, these representations provide a transparent pathway for students to understand how each transformation affects the mapping of function points and how the resulting rules emerge naturally from the underlying mathematics.

Taken as a whole, the findings from this work contribute to a clearer and more unified understanding of function transformations in secondary mathematics. They highlight a pathway for teachers to support reasoning-based instruction that helps students move beyond fragmented procedures toward a robust and transferable grasp of functional behavior. More broadly, this work underscores the value of designing transformation instruction that encourages learners to generalize, justify, and explore—practices essential to long-term mathematical proficiency. By clarifying the mathematical structure underlying function transformations and by offering tools that support deeper reasoning for learners, this work contributes to a more conceptually coherent and pedagogically effective treatment of a foundational topic in high school mathematics.

Future research may extend this framework in several productive directions. Empirical studies could examine how students' conceptual understanding evolves when instruction emphasizes derivation and reasoning rather than memorization. The same framework may also be adapted for more advanced topics—such as piecewise functions, transformations in the complex plane, matrix-based transformations in linear algebra, or Fourier transformation for digital signal processing—to explore its scalability across mathematical domains. Additionally, partnerships with classroom teachers could inform the development of instructional materials, dynamic software activities, or assessment tasks that further operationalize this approach in diverse learning settings.

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