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On Noether's Theorem and Applications in Classical Mechanics and Quantum Field Theory

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ABSTRACT: The role symmetries play in the laws of physics is explored in these papers. It covers systems from Newtonian mechanics to modern physics, such as the Standard Model. The math involved focuses on Lagrangian, Hamiltonian, and calculus of variations, which are foundational to understanding Noether's theorem. This paper uses the theorem to show how continuous symmetries lead to conservation laws, including conservation of momentum, angular momentum, energy, and charge. The paper then advances to topics such as gauge symmetry, complex scalar fields, and scalar quantum electrodynamics (QED). The paper emphasizes how symmetry provides a unifying framework for physics and its applications across classical and modern physics.
KEYWORDS: Physics and Astronomy, Theoretical and Computational and Quantum Physics, Noether's Theorem, Symmetry in Physics, Conservation Laws, Lagrangian Mechanics, Quantum Field Theory, Gauge Symmetry.

Introduction

The universe we observe works under many complex laws, from the conservation of energy and momentum to more complex theories such as general relativity and the standard model of particle physics. However, as complex as these laws might seem, they are rooted in a more fundamental concept of symmetry. Symmetries in physics are transformations that leave certain properties of a system unchanged.[1,2]

After Sir Isaac Newton formulated his Principia Mathematica, people worldwide began to study physics within the framework of Newtonian mechanics. This approach looked at nature in terms of forces and acceleration, which are mathematically described as vectors—abstract mathematical entities representing both magnitude and direction. Newton described kinematics and dynamics in terms of quantities that we now represent as vectors and laid the groundwork for what later became vector calculus.[3,4] While Newton's classical physics framework was influential, it overcomplicated certain systems, such as the double pendulum, which involves 5 vectors. Joseph Louis Lagrange proposed a different method for these types of systems.[5]

Lagrange came up with Lagrangian mechanics and found that nature always follows a path of least action. The actions are the integral of a quantity called the Lagrangian. There was no method to analyze which action was the least without computing all the integrals.[6,7] So, Lagrange, along with many other mathematicians, especially Leonhard Euler, developed the calculus of variations and derived the Euler-Lagrange equation.[8,9] A mathematician, Emmy Noether, expanded on the Euler-Lagrange equation and formulated Noether's theorem. The theorem states that for every continuous symmetry of a physical system, there exists a conserved quantity.[10,11] For example, a perfect sphere is continuously symmetric under rotational translation. If you suspected a symmetry in a system, you could use Noether's procedure and derive a conservation law. The three most common symmetries applied to Noether's theorem are:

1. Translational Symmetry in Space: Leads to the conservation of momentum.

2. Rotational Symmetry in Space: Leads to the conservation of angular momentum, as shown in Figure 1.

3. Translational Symmetry in Time: Leads to the conservation of energy.[12]

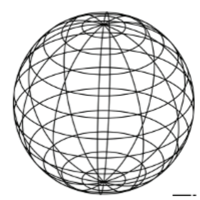

Figure 1: A sphere's rotational symmetry leads to angular momentum conservation via Noether's theorem.

The reason for analyzing Conservation laws is that they are among the most important tools in physics. They are extremely fundamental and allow for a more efficient method to solve complex physics problems.[13]

The question this paper addresses is to what extent Noether's theorem can be applied. As the paper will demonstrate, the principles of symmetry and conservation laws have applications across classical and modern physics. The theorem provides a unified framework that describes the behavior of many physical systems, from the conservation of momentum, angular momentum, and energy to even more complicated systems with gauge symmetry, complex scalar fields, and scalar quantum electrodynamics (QED).

Mathematical Prerequisites:

Before diving into the derivation, some mathematical prerequisites are needed. Some mathematics behind Lagrangian

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mechanics, Hamiltonian mechanics, and Noether's procedure is needed to fully understand the derivation.

Calculus of variations: The calculus of variations is a branch of mathematics that analyzes extrema. In regular calculus, extreme points are found by taking the derivative and setting it equal to 0. However, with the calculus of variations, instead of analyzing functions, we analyze functionals, which are functions of functions. To find the extreme functionals, we need to solve a differential equation and can't simply set the derivative equal to 0. Euler and Lagrange found out that the differential equation allows us to find extreme points. For physics purposes, it shows the path of least action.

So, we are trying to find a function that $y(x)$ makes a given functional $J[y]$ stationary. This function refers to the action in physics. The action is given by:

$$J[y] = \int_a^b F(x, y, y')dx \tag{1.1}$$

Where $y = y(x)$ is the function to be found, $y' = \frac{dy}{dx}$ and F is a given function of x, y, and y'

<u>Derivation of Euler-Lagrange Equation (Figure 2)</u>

Perturbation: Consider a small perturbation of the function $y(x)$ a small parameter ϵ and a function that $\eta(x)$ vanishes at the boundaries and b

$$y(x) \rightarrow y(x) + \epsilon\eta(x) \tag{1.2}$$

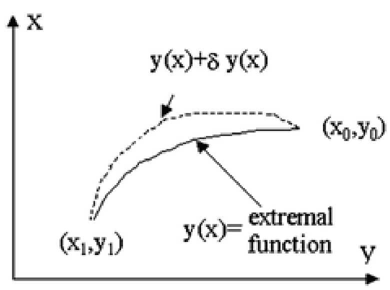

Figure 2: Visualization of the variation of a function $y(x)$ perturbed by $\epsilon\eta(x)$, which vanishes at the endpoints (x_1, y_1) and (x_0, y_0). This illustrates the idea of varying a path to find the one that minimizes the action in the calculus of variations.

Functional Variation: The functional $J[y]$ becomes:

$$J[y + \epsilon\eta] = \int_a^b F(x, y + \epsilon\eta, y' + \epsilon\eta')dx \tag{1.3}$$

First Variation: Expanding $J[y + \epsilon\eta]$ in the Taylor series and keeping terms up to the first order in:

$$\delta J = \frac{d}{d\epsilon} J[y + \epsilon\eta] \tag{1.4}$$

$$\delta J = \int_a^n \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)dx \tag{1.5}$$

Integration by Parts: Integrate the term involving η' parts, assuming

$$\eta(a) = \eta(b) = 0 \tag{1.6}$$

$$\delta J = \int_a^b \left(\frac{\partial F}{\partial y}\eta - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta\right)dx \tag{1.7}$$

$$\delta J = \int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)dx \tag{1.8}$$

Stationarity Condition: For δJ to be zero for all $\eta(x)$ the integrand must be zero:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{dy'}\right) = 0 \tag{1.9}$$

With this, we have understood the math of calculus of variations. This is the groundwork for the theory of Lagrangian mechanics. In Lagrangian mechanics, instead of forces, we analyze energy. The Lagrangian is equal to K–P, where K is the kinetic energy and P is the potential energy. While it doesn't have a simple physical interpretation and doesn't correspond directly to a measurable quantity like energy or momentum, it plays a central role in determining the dynamics of a system. What Lagrange found out was that nature always follows a path of least action, always trying to minimize something, which is the action. The action is:

$$S[q, t_2, t_1] = \int_{t_1}^{t_2} L(q, \dot{q})\, dt \tag{1.10}$$

Methods: Noether's procedure

To see the full extent how Noether's theorem can be applied, we must first analyze Noether's procedure

1. **Identifying the action S and the Lagrangian L.**
2. **Determining the symmetry transformation** $q_i \rightarrow q_i + \epsilon\eta_i$
3. **Calculating the variation of the action** under this transformation.
4. **Using the Euler-Lagrange equation** to simplify the expression.
5. **Integrating by parts** to isolate the boundary terms.
6. ***Identifying the conserved quantity Q.***

Q here represents the conserved quantity associated with a given symmetry. Depending on the symmetry, Q may represent energy, linear momentum, angular momentum, or another conserved charge. Noether showed that if the Lagrangian remains unchanged under a continuous transformation, this invariance leads directly to the conservation of some physical quantity.

The paper will look at different systems and try to apply Noether's procedure to each case to obtain the Noether current for each.

For some of the simpler, classical systems, we will justify how the Lagrangian and the action are derived. However, for more complex systems later in the paper, especially dealing with quantum field theory, the justification will not be provided and referenced to existing papers.

Results: Noether's theorem applied to Systems

2.1: Energy Conservation due to Noether's theorem:

Consider a general system L. This system could be anything from a simple pendulum to a complex multi-particle system. The system is defined by a set of generalized coordinates q_i and their corresponding $\dot{q}_i$ velocities. We assume that L has no explicit dependence on time. The dynamics of the system are governed by a Lagrangian $L(q,\dot{q})$

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Since L depends on the evolving functions $q_i(t)$ and $\dot{q}_i(t)$, the total derivative with respect to time can be computed in two ways. The first is the chain rule:

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{2.1}$$

The second way is viewing L as a function of time through $q(t)$, $\dot{q}(t)$. This derivative is simply dL / dt

These two perspectives must agree. The point to notice is that there isn't a $\partial L/\partial t$ term, because L has no explicit time dependence.

To reconcile the two expressions, we replace $\partial L/\partial q_i$ using the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2.2}$$

Substituting this into (2.1) gives:

$$\frac{dL}{dt} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{2.3}$$

We can now reorganize (2.3), noticing that:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) \tag{2.4}$$

Therefore,

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right) \tag{2.5}$$

Rearranging, we find:

$$\frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L\right) = 0 \tag{2.6}$$

Equation (2.6) shows that the quantity:

$$H = \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L \tag{2.7}$$

Is conserved in time. This quantity is called the Hamiltonian of the system.

In the usual mechanical case where L = T – V, with kinetic and Potential energy, we find explicitly:

$$\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} = 2T \tag{2.8}$$

So that

$$H = 2T - L = 2T - (T - V) = T + V \tag{2.9}$$

Thus, the Hamiltonian corresponds to the total energy of the system: the sum of the kinetic and potential energy.

The conservation of energy is directly linked to the invariance of the system under time translations. If the Lagrangian doesn't change with time, then the total energy stays constant. This shows the fundamental principle that the outcome of a process doesn't depend on when it takes place because the laws of physics are time invariant. Energy can't be created or destroyed. It can only change form between Kinetic and Potential.

2.2: Conservation of momentum:

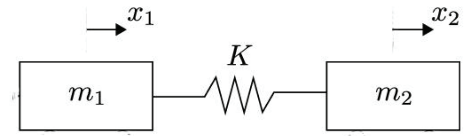

Figure 3: Visualization of a two-mass spring system undergoing a uniform spatial translation by a constant x_1 and x_2, which are equal.

Consider this system of 2 springs: The Lagrangian of this system (Figure 3) is:

$$L = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\dot{x}^2 - \tfrac{1}{2}k(x_1 - x_2)^2 \tag{2.10}$$

Now we apply Space translation to this system: $x_2{}' \rightarrow x_2{}' + c$

Now with this translation, we can calculate the Lagrangian again:

$$L(x') = \tfrac{1}{2}m_1(x_1 \dot{+} c)^2 + \tfrac{1}{2}m_2(x_2 \dot{+} c)^2 - \tfrac{1}{2}k(x_1 + c - x_2 - c)^2 \tag{2.11}$$

The derivative of a constant is just 0, and the c's in the potential energy cancel out, leaving the original Lagrangian. L=L' So, we have symmetry.

<u>Now to apply Noether's procedure:</u>

Let $\bar{x}_1(t)$ and $\bar{x}_2(t)$ be the true paths of the masses. Then consider a tiny time-dependent variation.

$$\overline{x}_1(t) = \overline{x}_1(t) + \varepsilon_1(t) \tag{2.12}$$

$$\overline{x}_2(t) = \overline{x}_2(t) + \varepsilon_2(t) \tag{2.13}$$

Note that $\varepsilon_1 = \varepsilon_2 = 0$ since the variant is done equally for both. The action without the variation is:

$$S[x_1(t),\, x_2(t)] = \int_{t_1}^{t_2} L(x(t),\, \dot{x}(t))\, dt \tag{2.14}$$

The action with the variation should have 3 terms: the original action, the variational action, and some variation of second order:

$$S[x_1(t) + \varepsilon_1(t),\, x_2(t) + \varepsilon_2(t)] = S[x_1(t),\, x_2(t)] + \delta S + O(\varepsilon^2) \tag{2.15}$$

$$= \int_{t_1}^{t_2} \tfrac{1}{2}m_1(\dot{\overline{x_1}} + \dot{\varepsilon})^2 + \tfrac{1}{2}m_2(\dot{\overline{x_2}} + \dot{\varepsilon})^2 - \tfrac{1}{2}k(\overline{x_1} + \varepsilon - \overline{x_2} - \varepsilon)^2 \tag{2.16}$$

$$= \int_{t_1}^{t_2} \tfrac{1}{2}m_1\left(\dot{\overline{x_1}}^2 + 2\,\dot{\overline{x_1}}\,\dot{\varepsilon} + \dot{\varepsilon}^2\right) + \tfrac{1}{2}m_2\left(\dot{\overline{x_2}}^2 + 2\,\dot{\overline{x_2}}\,\dot{\varepsilon} + \dot{\varepsilon}^2\right) - \tfrac{1}{2}k(\overline{x_1} - \overline{x_2})^2 \tag{2.17}$$

The $\dot{\varepsilon}^2$ is too small of a variation, so we can ignore it. Moreover, notice that we can separate this integral into the original action and the $\dot{\varepsilon}^2$ terms.

$$= \int_{t_1}^{t_2}\left(\tfrac{1}{2}m_1\,\dot{\overline{x_1}}^2 + \tfrac{1}{2}m_2\,\dot{\overline{x_2}}^2 - \tfrac{1}{2}k(\overline{x_1} - \overline{x_2})^2\right) dt + \left|\int_{t_1}^{t_2}\left(m_1\,\dot{\overline{x_1}}\dot{\varepsilon} + m_2\,\dot{\overline{x_2}}\dot{\varepsilon}\right) dt\right| + O(\varepsilon^2) \tag{2.18}$$

The only integral within the vertical bars is the variational action, and the one we care about. Then, by the principle of least action, δS = 0 and the endpoints $\varepsilon_1(t)$ = $\varepsilon_2(t)$

$$\int_{t_1}^{t_2}\left(m_1\,\dot{\overline{x_1}}\,\dot{\varepsilon} + m_2\,\dot{\overline{x_2}}\,\dot{\varepsilon}\right) dt = 0 \tag{2.19}$$

Next, integration by parts is applied:

$$\int_{t_1}^{t_2}\left(m_1\,\dot{\overline{x_1}}\dot{\varepsilon} + m_2\,\dot{\overline{x_2}}\dot{\varepsilon}\right) dt = m_1\,\dot{\overline{x_1}}\,\varepsilon + m_2\,\{\dot{\overline{x_2}}\,\varepsilon \vdash_{t1}^{t_2}| - \int_{t_1}^{t_2}\epsilon\left(m_1\,\frac{d}{dt}(\dot{\overline{x_1}}) + m_2\,\frac{d}{dt}(\dot{\overline{x_2}})\right) dt \tag{2.20}$$

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Remember, $\varepsilon(t_1) = \varepsilon(t_2) = 0$.So, the first term cancels out. Therefore:

$$0 = \int_{t_1}^{t_2} \varepsilon \left(m_1 \frac{d}{dt}(\dot{x_1}) + m_2 \frac{d}{dt}(\dot{x_2})\right) \tag{2.21}$$

The only way this integral is 0 is if the integrand is 0.

$$0 = \frac{d}{dt}\left(m_1 (\dot{x_1}) + m_2 (\dot{x_2})\right) \Rightarrow \frac{d}{dt}(p_1 + p_2) = 0 \tag{2.22}$$

We have shown that the change in momentum of the 2 masses stays constant and thereby shows conservation of momentum.

2.3: Conservation of Angular Momentum:

Consider the system shown in Figure 4. Let the sun be m_1 and the Earth be m_2. We make the approximation $m_1 >> m_2$ so the Sun can be treated as fixed at the origin. Therefore, the Lagrangian of this system should be:

$$L = \frac{1}{2}m_2(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{G_{m_1 m_2}}{r} \tag{2.23}$$

There are 2 parameters that we can change in this problem. r or θ. If we change r, the P.E changes, and so does the Lagrangian. So, to have symmetry, only can θ change. $\theta' \rightarrow \theta + c$. Where c is constant. Since $\dot{\theta}' = \dot{\theta}$ and r is unchanged, the Lagrangian doesn't change. $L=L'$

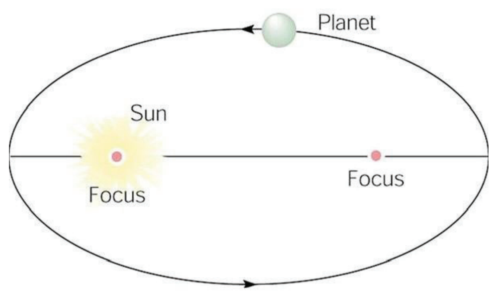

Figure 4: Visualization of the Earth-Sun system, where the Earth (mass m2) orbits the Sun (mass m1) in an elliptical path, with the Sun assumed fixed.

Consider a tiny time-dependent rotational variation. $\overline{\theta} \rightarrow \overline{\theta} + \varepsilon(t)$.The action after the variations gets separated by 3 terms. The original action, the variational action, and some variation of the second order.

$$S[\overline{\theta} + \varepsilon, r] = S[\overline{\theta}, r] + \delta S + O(\varepsilon^2) \tag{2.24}$$

$$= \int_{t_1}^{t_2} \left[\frac{1}{2}m_2\left(\dot{r}^2 + r^2\left(\dot{\overline{\theta}} + \dot{\varepsilon}\right)^2\right) + \frac{G_{m_1}m_2}{r}\right] dt \tag{2.25}$$

$$= \int_{t_1}^{t_2} \left[\frac{1}{2}m_2\left(\dot{r}^2 + r^2\left(\dot{\overline{\theta}}^2 + 2\dot{\overline{\theta}}\dot{\varepsilon} + \dot{\varepsilon}^2\right)\right) + \frac{G_{m_1}m_2}{r}\right] dt \tag{2.26}$$

The ε^2 is too small of a variation, so we can ignore it. Moreover, notice that we can separate this integral into the original action and the terms.

$$= \int_{t_1}^{t_2} \left[\frac{1}{2}m_2\left(\dot{r}^2 + r^2\dot{\overline{\theta}}^2\right) + \frac{G_{m_1}m_2}{r}\right] dt + \left|\int_{t_1}^{t_2} m_2 r^2 \dot{\overline{\theta}}\dot{\varepsilon}\, dt\right| + \int_{t_1}^{t_2} \frac{1}{2} m_2 r^2 \dot{\varepsilon}^2\, dt \tag{2.27}$$

The only integral within the vertical bars is the variational action, and the one we care about. Then by the principle of least action, $\delta S = 0$ and the end points $\varepsilon_1 = \varepsilon_2 = 0$

$$0 = \int_{t_1}^{t_2} \varepsilon \frac{d}{dt}\left(m_2 r^2 \dot{\overline{\theta}}\right) dt \tag{2.28}$$

After integration by parts:

$$\int_{t_1}^{t_2}\left(m_2 r^2 \dot{\overline{\theta}} \dot{\varepsilon}\right) dt = \varepsilon m_2 \left[r^2 \dot{\overline{\theta}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \varepsilon \frac{d}{dt}\left(m_2 r^2 \dot{\overline{\theta}}\right) dt \tag{2.29}$$

Notice the first term cancels out because of the boundary conditions. Therefore:

$$0 = \int_{t_1}^{t_2}\left(m_2 r^2 \dot{\overline{\theta}} \dot{\varepsilon}\right) dt \tag{2.30}$$

The only way this integral is 0, is if the integrand is 0.

$$\frac{d}{dt}\left(m_2 r^2 \dot{\overline{\theta}}\right) = 0 \tag{2.31}$$

It is convenient to define the angular momentum as $J = mr^2\dot{\theta}$. Then the equation becomes:

$$\frac{d}{dt}J = 0 \tag{2.32}$$

We have shown that the change in the angular momentum of the Earth and the Sun stays constant. Showing conservation of angular momentum.

2.4: Conservation of mass-energy:

For the derivation of the mass energy equivalence equation, we are going to assume it's a relativistic free particle. By free particle, it means that there is no P.E., or that there can be no force on the particle, thereby the momentum remains constant. The particle is only moving through a vacuum. Moreover, by relativity, the only main assumption we are making is the postulates of Special relativity.

1st postulate: Laws of physics are the same and can be stated in their simplest form in all inertial frames of reference

2nd postulate: speed of light c is a constant, independent of the relative motion of the source.

Mathematically, all this will do is put a gamma term (γ) in the equations.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.33}$$

<u>Philosophical Assumptions:</u> When dealing with free particles in physics, understanding the concept of travel through space and the importance of the Lagrangian is crucial. The Lagrangian explains all the physics about a system in a philosophical manner. It has all the physical properties relevant to understanding a system. This includes its motion and the factors influencing that motion. In physics, only specific properties, such as mass, acceleration, and velocity, significantly impact a system's motion or its interactions. Other properties, like color or luster, are not as important. Essentially, the Lagrangian can be thought of as the energy analogy to the equation F=ma.

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In special relativity, a particle isn't only travelling through space and time, but instead through space-time. The path the particle takes through spacetime is called the world line, as shown in Figure 5.

The world line of a particle is given by its spacetime coordinates: $(ct, x(t), y(t), z(t)$. There is ct, instead of just t, because spacetime is a four-dimensional continuum where time and space coordinates are combined into a single entity called the spacetime interval.

$$S = \sqrt{c^2t^2 - x^2 - y^2 - z^2} \tag{2.34}$$

If we instead consider a small arc length ds, the length becomes:

$$ds = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2} \tag{2.35}$$

We need to introduce a concept called proper time. Proper time is the time interval measured by a clock moving with constant velocity from one event to another. An observer in motion relative to a clock will always observe it running slower than a clock at rest in their own frame. Proper time (τ) is specifically the time read by a clock present at both events, with both events occurring at the same place in the clock's rest frame.

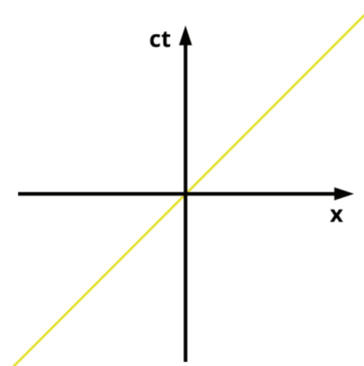

Figure 5: *Visualization of a world line of a relativistic free particle traveling through spacetime.* This illustrates the motion of a particle not just through space, but through four-dimensional spacetime, where the arc length of the world line corresponds to the proper time experienced by the particle.

Proper time is related to the space-time interval (s) between two time-like events by the equation:

$$\Delta\tau = \frac{\Delta s}{c} \Longrightarrow d\tau = \frac{ds}{c} \tag{2.36}$$

From the arc length equation, plug into the proper time equation.

$$d\tau = \frac{\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}}{c} \tag{2.37}$$

We can define a velocity for dx, dy, and dz, to get:

$$d\tau + \frac{\sqrt{c^2dt^2 - (v\,dt)^2}}{c} \Longrightarrow d\tau = \frac{\sqrt{dt^2\,(c^2 - v^2)}}{c} \Longrightarrow d\tau = \frac{dt\sqrt{(c^2 - v^2)}}{c} \Longrightarrow d\tau = \frac{dt \cdot c\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}{c} \tag{2.38}$$

Simplifying further, and using the Lorentz factor, we get the equation:

$$d\tau = dt\sqrt{\left(1 - \frac{v^2}{c^2}\right)} \Longrightarrow d\tau = \frac{dt}{\gamma} \tag{2.39}$$

Lastly, we need to make one assumption. The Lagrangian must be Lorentz invariant. The obvious invariance is the length of the world line. Since it's a free particle, we can just let the action be:

$$S = \alpha \int_a^b ds \tag{2.40}$$

Here, α is some constant. This just means the action is proportional to the length of the world line. The length of the world line is also equal to the proper time interval. Then we can substitute the proper time equation derived earlier.

$$S = \alpha \int_{\tau_1}^{\tau_2} d\tau \Longrightarrow S = \alpha \int_{\tau_1}^{\tau_2} \frac{dt}{\gamma} = \alpha \int_{\tau_1}^{\tau_2} \left(\sqrt{1 - \frac{v^2}{c^2}}\right) dt \tag{2.41}$$

To get the value of γ , we can keep in mind that in the non-relativistic limit (v<<c), the canonical momentum defined by $dL/d\dot{q}$ reduces to the classical expression mv.

$$mv = \frac{\partial L}{\partial v} = \left(\frac{1}{2}\right) - \frac{2\alpha c^{-2}v}{\frac{\sqrt{1-v^2}}{c}} = -\gamma\,\alpha\,c^{-2}v \tag{2.42}$$

For the variations to match, $\alpha = -mc^2$ so now the relativistic Lagrangian is:

$$L = -\frac{mc^2}{\gamma} \tag{2.43}$$

Derivation: Now we can apply Noether's theorem to the Lagrangian.

Spacetime Translation Symmetry:

Time Translation Symmetry: The Lagrangian (L) does not depend explicitly on time (t), implying conservation of energy. For time translational symmetry, Noether's theorem states that the conserved quantity is the total energy. E = $\partial L/\partial t$. This expression can be justified, but needs more discussion.

Space Translation Symmetry: The Lagrangian (L) does not depend explicitly on position (r), implying conservation of momentum. For space translation symmetry, the conserved quantity is the momentum P. P = $\partial L/\partial \dot{r}$

Calculating P first.

$$p_i = \frac{\partial L}{\partial \dot{x}} = \frac{(m \cdot c \cdot \dot{x})}{\sqrt{1 - \frac{v^2}{c}}} \tag{2.44}$$

The Hamiltonian, which is H = p$\dot{x}$ - L becomes:

$$H = \frac{(m \cdot c \cdot \dot{x})}{\sqrt{1 - \frac{v^2}{c}}} + \frac{(mc^2)}{\sqrt{1 - \frac{v^2}{c}}} = \frac{m(c^2 \cdot v^2)}{\sqrt{1 - \frac{v^2}{c}}} \tag{2.45}$$

Remember that the Hamiltonian (H) just shows the total energy of the system. Therefore:

$$E = \frac{m(c^2 \cdot v^2)}{\sqrt{1 - \frac{v^2}{c}}} \tag{2.46}$$

For a particle that isn't moving (v = 0), the equation gets reduced to

$$E = mc^2 \tag{2.47}$$

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3: Noether's Theorem Applied in Reverse:

In the previous systems, we applied the Noether theorem to particles. Now, will apply Noether's theorem to fields, specifically the electromagnetic field. The transition from examining symmetries of particles to the field is a big shift. We can't just consider fields as just functions of time, but as functions of both space and time. To do that, we need to introduce the mathematical concept of tensors.

To better understand the difference between Lagrangian mechanics in classical vs field theory, is to think of an analogy of a car traveling along a road. The Lagrangian allows us to describe the car's motion by incorporating its speed and position on the road. This is how we treat particles. But with fields, we instead think of multiple cars along the road, and each segment of the road can have unique characteristics and dynamics, and this is where the idea of Lagrangian density becomes important because they are functions of both space and time.

Think of each segment of the road as representing a point in space and time.

(Φ) could represent the number of cars or their speed at each point on the road. Aka. Traffic flow.

($\partial_\mu \Phi$) represents how the number of cars or their speed changes from one point on the road to another. Aka. Changes in the traffic.

So, the Lagrangian density describes the traffic dynamics over the entire road. It shows how each segment of the road interacts with its neighbors in space and time.

Another shift we will be considering, finding the true usefulness of Noether's theorem, is to instead examine how the existence of conservation laws leads to conserved quantities. We will be analyzing the conservation of electric charge and how it gives rise to a symmetry, specifically gauge symmetry.

Electromagnetic Field: The electromagnetic field is described by 4 equations, namely the Maxwell equations.

1. $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ -- Gauss's Law for Electricity
2. $\nabla \cdot B = 0$ -- Gauss's Law for Magnetism
3. $\nabla \times E = -\frac{\partial B}{dt}$ -- Faraday's Law
4. $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ -- Ampère's Law

Another way to describe this is to use the Electromagnetic Tensor. Which gives us the benefit of being invariant of the coordinate system we use. The Electromagnetic Tensor is defined as:

$$F_{\mu v} = \partial_\mu A_v - \partial_v A_\mu \tag{3.1}$$

Some understanding of the notation here is necessary before proceeding.

• A_μ represents the components of the electromagnetic potential,

• ∂_μ denotes the partial derivative with respect to the spacetime coordinate, x^μ

• μ, v are indices running from 0 to 3, corresponding to the spacetime dimensions (time and spatial dimensions).

Conservation of charge: In any closed system, the sum of all positive and negative charges remains unchanged.

Consider Figure 6. The total charge inside is found by integrating the charge density over the volume:

$$Q(t) = \int_V \rho(r,t)\ d^3r \tag{3.2}$$

The change in Q over time is due to the flow of charge across the boundary surface B of V. Now, we introduce another variable, current density (J). This is a vector representing the amount of charge per unit area per unit time flowing across S.

Consider a patch on B at point r with are *dA* The current through this patch is given by the component of J perpendicular to the surface, where n is the unit vector normal to the surface: *J . n*

Current through the patch = *J . ndA*

Integrating this over the entire surface, we get,

$$I = \int_B J \cdot n\, dA \tag{3.3}$$

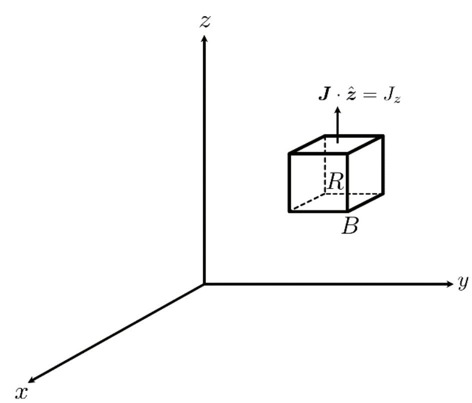

Figure 5: Volume *V* in space with charge density representing the charge per unit volume at point r and time t.

Now we can state the conservation of charge mathematically. (I) measures the amount of charge per unit time leaving the box (or entering it, if [I] came out negative). Local conservation of charge is the statement that if charge (I) per unit time flows out through the boundary, then the amount of charge Q inside the volume of the box goes down at that same rate:

$$\frac{dQ}{dt} = -I \tag{3.4}$$

The minus sign reflects our convention that I > 0 means outward flow. To convert the surface integral in (3.3) into a volume integral, we need to apply Gauss's theorem.

$$\oint_B J \cdot n\, dA = \int_V \nabla \cdot J\, d^3r \tag{3.5}$$

Substituting (3.5) into (3.4) and using $Q(t) = \int_V \rho\, d^3r$, we get:

$$\frac{d}{dt}\int_R \rho\ d^3r = -\int_B J \cdot n\, dA \tag{3.5}$$

To encompass *all* of space, so that the boundary is going to infinity, the current density (J) should go to zero in any physically reasonable setup, since there's nowhere left for the current to flow out to. Then the right-hand side vanishes, and this equation says that the total charge in all of space is constant.

Since this equation holds for any arbitrary volume (V), the integrands must be equal pointwise, leading to the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 = \partial_0 \rho + \nabla \cdot J \tag{3.6}$$

The continuity equation in four-dimensional spacetime is:

$$\partial_\mu j^\mu = 0 \tag{3.7}$$

This depicts conservation of charge. Now we will go back to the electromagnetic field tensor, which is a key concept in relativistic electromagnetism. It compactly encapsulates the electric and magnetic fields.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.8}$$

Where A^μ is the four-potential, which includes the scalar Φ potential and the vector potential A

$$A^\mu = (\phi, A) \tag{3.9}$$

Now, to derive conservation of charge from the symmetry, we need the Lagrangian.

The Lagrangian density for the free electromagnetic field is given by:

$$L_{EM} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.10}$$

The coupling of the electromagnetic field to charged particles is introduced through the current density J^μ and the four-potential A_μ. The coupling term in the Lagrangian density is:

$$L_{int} = -j^\mu A_\mu \tag{3.11}$$

The total Lagrangian density L is the sum of the electromagnetic field Lagrangian and the interaction term.

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \tag{3.12}$$

Now we perform a symmetrical operation that leaves the Lagrangian invariant. Such an operation or transformation is called the Gauge transformation.

$$A'_\mu = A_\mu + \partial_\mu \alpha \tag{3.13}$$

α is some spacetime scalar function. Let's now see how the Electromagnetic Tensor changes.

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu \tag{3.14}$$

Substituting the transformed potential A_μ':

$$F'_{\mu\nu} = \partial_\mu(A_\nu + \partial_\nu \alpha) - \partial_\nu(A_\mu + \partial_\mu \alpha) = \partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha - \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha \tag{3.15}$$

Since the mixed partial derivatives are symmetric,

$$\partial_\mu \partial_\nu \alpha = \partial_\nu \partial_\mu \alpha \tag{3.16}$$

These terms cancel out, leaving us with:

$$F'_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{3.17}$$

So, the field tensor is invariant under the gauge transformation. So, the Lagrangian density should also be invariant. $L = L'$.

Noether's Procedure:

Now let's perform Noether's procedure with the gauge transformation. Since the Electromagnetic Tensor was invariant under Gauge transformation, the field term $L_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = 0$

So, its variation in the action does not change. Let's look at the coupling terms. $L = -j^\mu A_\mu$

Applying a variation, we get: $\delta L = -j^\mu \,\partial A_\mu = -j^\mu A_\mu - -j^\mu \,\partial_\mu \alpha$.

$$\delta S = \int_{t_1}^{t_2} \delta L \; d^4x = - \int_{t_1}^{t_2} \left[j^\mu A_\mu + j^\mu \partial_\mu \alpha \right] d^4x \tag{3.18}$$

$\int_{t_1}^{t_2} j^\mu A_\mu$ is just the original action of the Lagrangian, which is $\Rightarrow -\int_{t_1}^{t_2} j^\mu A_\mu \, d^4x$ We now perform Integration by parts.

$$\Rightarrow -\int_{t_1}^{t_2} j^\mu A_\mu \, d^4x = \left[j^\mu A_\mu \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} j^\mu \,\partial_\mu \,\alpha \, d^4x \tag{3.19}$$

$\alpha(t_1) = \alpha(t_2) = 0$. From the conservation of charge, we have:

$$\partial\mu J^u = 0 \Rightarrow \int_{t_1}^{t_2} \alpha(\partial\mu J^\mu) \, d^4x \tag{3.20}$$

Because of the current conservation ($\partial_\mu J^\mu$), this vanishes and $\delta S = 0$. Showing the Gauge symmetry. But there are some important implications. We assumed that conservation of charge had to exist for there to be Gauge symmetry. We didn't get any conserved quantity. This is because we had a redundancy of information. This means that all components of a field tensor or set of equations are not independent. Some can be derived from others due to symmetries or constraints. This redundancy ensures that physical principles, like the conservation of electric charge, are naturally satisfied and can't be derived directly from Noether's theorem.

4: Noether's theorem to QFT:

Symmetries and Noether's theorem can also be applied to quantum fields. In quantum field theory, every type of particle is associated with a corresponding quantum field. For example, the electromagnetic field is associated with photons, while the electron field is associated with electrons. The specific field we will be looking at is the complex scalar field.

4.1: Complex scalar field:

Unlike the electromagnetic field tensor, which describes both the electric and magnetic fields, a complex scalar field is a type of quantum field characterized by values that are complex numbers (numbers that have both real and imaginary parts) at each point in space and time. An example of a complex scalar field is the Higgs Field. The Lagrangian density of a complex scalar field:

$$L = (\partial_\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi \tag{4.1}$$

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We now perform a global U (1) phase transformation.
$\Phi \rightarrow \Phi' = e^{i\beta}\Phi,\ \Phi^* \rightarrow \Phi^{*'} = e^{i\beta}\Phi^{*'}$

This changes the phase of the field across all spacetime points. In the transformation β, is a constant function and $e^{i\beta}$ is a complex number that changes the phase of $\Phi(x)$ and $\Phi^*(x)$. To see the Lagrangian is symmetric under this transformation, first let's look at the kinetic energy term:

$$\partial_\mu \phi \rightarrow \partial_\mu \phi' = \partial_\mu\left(e^{i\beta}\phi\right) = e^{i\beta}\partial_\mu \phi \quad (4.2)$$

$$\partial_\mu \phi^* \rightarrow \partial_\mu \phi^{*'} = \partial_\mu\left(e^{-i\beta}\phi^*\right) = e^{-i\beta}\partial_\mu \phi^* \quad (4.3)$$

The product of these two terms becomes:

$$(\partial_\mu \phi^*)(\partial_\mu \phi) \rightarrow \left(e^{-i\beta}\partial_\mu \phi^*\right)\left(e^{i\beta}\partial_\mu \phi\right) = 1 \cdot \partial_\mu \phi^* \partial_\mu \phi \quad (4.4)$$

This is because the phase factors $e^{-i\beta}$ and $e^{i\beta}$ α cancel each other out, leaving the kinetic term unchanged. To see why this is, we'll analyze Euler's formula.

$$e^{i\beta} = \cos(\beta) + i\sin(\beta) \text{ and } e^{-i\beta} = \cos(-\beta) + i\sin(-\beta) = \cos(\beta) - i\sin(\beta) \quad (4.5)$$

Multiplying them together, we get.

$$e^{i\beta}\cdot e^{-i\beta} = [\cos(\beta) + i\sin(\beta)][\cos(\beta) - i\sin(\beta)] = \cos^2(\beta) - i\cos(\beta)\sin(\beta) + i\sin(\beta)\cos(\beta) - (i\sin(\beta))^2 = \cos^2(\beta) - (i\sin(\beta))^2$$

and since

$$i^2 = -1, \Longrightarrow \ = \cos^2(\beta) + \sin^2(\beta) = 1 \quad (4.6)$$

Now let's analyze the mass term.

$$\phi^*\phi \rightarrow \phi'^*\phi' = \left(e^{-i\beta}\phi^*\right)\left(e^{i\beta}\phi\right) = \phi^*\phi \quad (4.7)$$

Since the 2 phases cancel out, we get back the original Φ terms. The mass will stay constant, and therefore the mass term is also invariant, meaning the whole Lagrangian is invariant. $L = L'$.

<u>Noether's procedure:</u>

$$\delta[S] = \int_{t_1}^{t_2} \delta L \ d^4x \quad (4.8)$$

Where $\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \phi^*}\phi^* + \partial\frac{\partial L}{(\partial_\mu \phi)}\delta(\partial_\mu\phi) + \frac{\partial L}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi *)$ (4.9)

Now let's solve each term by term of the variation in the Lagrangian.

$$\frac{\partial L}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \partial^\mu\phi^* \quad \text{and} \quad \frac{\partial L}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*) = \partial^\mu\phi^* \quad (4.10)$$

$\delta\phi = \phi' - \phi = e^{i\beta}\phi - \phi$ We can expand $e^{i\beta}$ using Taylor's expansion.

$$e^{i\beta} = 1 + i\beta - \frac{\beta^2}{2!} + \cdots \Rightarrow e^{i\beta}(\phi) = \phi + i\beta\phi - \frac{\beta^2}{2!}\phi + \cdots \Rightarrow e^{i\beta}(\phi) - \phi = i\beta\phi - \frac{\beta^2}{2!}\phi + \cdots$$

We can neglect the higher-order terms and be left with $\delta\phi \approx i\beta\phi$ and $\delta\phi^* \approx -i\beta\phi^*$

Also notice that $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) = \partial_\mu(i\beta\phi)$ and $\delta(\partial_\mu\phi^*) = \partial_\mu(\delta\phi^*) = \partial_\mu(-i\beta\phi^*)$

Lastly, we have, $\frac{\partial L}{\partial\phi} = -m^2\phi^*$ and $\frac{\partial L}{\partial\phi^*} = -m^2\phi^*$. Now let's substitute these values into the variational Lagrangian.

$$\delta L = -m^2\phi^*\phi i\beta + m^2\phi\phi^* i\beta + \partial^\mu\phi^*\partial_\mu\phi i\beta - \partial^\mu\phi\partial_\mu\phi^* i\beta = i\beta\left(\phi^*\partial^\mu\partial_\mu\phi - \phi\partial^\mu\partial_\mu\phi^*\right)$$

$$\delta[S] = \int_{t_1}^{t_2}\delta L\ d^4x = \int_{t_1}^{t_2}\left(i\beta\phi^*\partial^\mu\partial_\mu\phi\right)d^4x - \int_{t_1}^{t_2}\left(i\beta\phi\partial^\mu\partial_\mu\phi^*\right)d^4x \quad (4.11)$$

We can now perform integration by parts on the first integral and plug in the endpoints.

$\int_{t_1}^{t_2}\left(i\beta\phi^*\partial^\mu\partial_\mu\phi\right)d^4x = i\left([\beta\partial_\mu]_{t_1}^{t_2} - \int_{t_1}^{t_2}\beta\phi^*\partial^\mu\phi\right)$. Since $\beta(t_1) = \beta(t_2) = 0$, we are left with:

$$-i\int_{t_1}^{t_2}\beta\phi^*\partial^\mu\phi \quad (4.12)$$

We can do the same for the second integral. Perform integration by parts and plug in endpoints.

$\int_{t_1}^{t_2}\left(i\beta\phi^*\partial^\mu\partial_\mu\phi\right)d^4x = -i\left([\beta\partial_\mu]_{t_1}^{t_2} - \int_{t_1}^{t_2}\beta\phi\partial^\mu\phi^*\right)$. since $\beta(t_1) = \beta(t_2) = 0$, we are left with:

$$i\int_{t_1}^{t_2}\beta\phi\partial^\mu\phi^* \quad (4.13)$$

Combining these two integrals together, we have,

$$i\int_{t_1}^{t_2}\beta[\partial^\mu\phi\phi^* - \partial^\mu\phi^*\phi]d^4x = 0 \quad (4.14)$$

By the principle of least action

$$\Longrightarrow i[\partial^\mu\phi\phi^* - \partial^\mu\phi^*\phi] = 0 \quad (4.15)$$

So, the conserved current is $J^\mu = i\,[\partial^\mu\Phi\Phi^* - \partial^\mu\Phi^*\Phi]$. This depicts the conservation of charge in QED.

What if β was instead a function of space and time. $\beta \rightarrow \alpha\ (x,\ t)$

$\Phi \rightarrow \Phi' = e^{i\alpha(x,t)}\Phi$ and $\Phi^* \rightarrow \Phi^{*'} = e^{-i\alpha(x,t)}\Phi^*$. Like before, let's analyze the K.E. term first.

$\partial_\mu\phi \rightarrow \partial_\mu\left(e^{i\alpha(x,t)}\phi\right) = e^{i\alpha(x,t)}\left(\partial_\mu\phi + i(\partial_\mu\alpha)\phi\right)$. We get this expression if we use Taylor's Expansion. We could do the same for $\partial_\mu\Phi^*$.

$$\partial_\mu\phi^* \rightarrow \partial_\mu\left(e^{i\alpha(x,t)}\phi^*\right) = e^{i\alpha(x,t)}\left(\partial_\mu\phi^* - i(\partial_\mu\alpha)\phi^*\right).\ ^{14} \quad (4.16)$$

Therefore, $L' = e^{i\alpha(x,t)}\cdot e^{-i\alpha(x,t)}\left(\partial_\mu\phi^* - i(\partial_\mu\alpha)\phi^*\right)\left(\partial_\mu\phi + i(\partial_\mu\alpha)\phi\right) - m^2\phi'\phi'^*$. Expanding, we get.

$$L' = \partial_\mu\phi^*\partial^{(\mu}\phi + i\phi^*(\partial_\mu\alpha)\partial^{(\mu}\phi - i\phi(\partial^\mu\alpha)\partial_\mu\phi^* - (\partial_\mu\alpha)(\partial^\mu\alpha) + (\phi^*\phi) - m^2\phi'\phi'^* \quad (4.17)$$

This shows that $L \neq L'$ because of the additional: $\partial_\mu\alpha$ terms. They don't vanish.

When the phase transformation was a function of space and time, the Lagrangian of the complex scalar wasn't invariant. This is called the local U (1) phase transformation. This transformation led to a symmetry that was internal. The phase of the complex scalar field can be locally altered (meaning it can be changed at each point independently) without affecting the overall physics. The specific conserved quantity of a complex scalar field depends on the context of the field and the physical theory being discussed. In some theories, complex scalar fields can represent particles with conserved quantum numbers like baryon number or lepton number. In other theoretical models, such as quantum electrodynamics (QED), a complex scalar field can represent particles with electric charge. It can also be applied to the Standard Model of particle physics, where the Higgs field is a complex scalar field.[15]

This invariance under the local phase changes is important in constructing theories like the Standard Model of particle physics. This also leads to the Higgs field and spontaneous

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symmetry breaking. The Higgs field has a symmetric potential, and the field value at each point is zero, which represents a high-energy, unstable state. When the field transitions to a lower-energy state where it has a non-zero magnitude everywhere in space, the specific direction of the field in the complex plane breaks the symmetry. Particles interacting with the Higgs field will then gain mass. The interaction depends on the field's magnitude, which is now non-zero and uniform across space.[16]

We will look at the complex scalar field in the context of QED.

4.2: Scalar QED:

Let's now apply Noether's theorem to scalar QED, which extends the principles of QED as it deals with spin-1/2 particles like electrons to scalar fields. Scalar QED is an extension of classical electrodynamics and quantum field theory. It describes the interaction between scalar fields (fields that are represented by scalar particles, which have spin zero) and the electromagnetic field.[17] The Lagrangian of the scalar QED is:

$$L = (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (4.18)$$

The mass term and the electromagnetic tensor have already been introduced before. The only new thing is the covariant derivative, incorporating the interaction with the gauge field A_μ.[18]

$$L = (\partial_\mu\phi^* + ieA_\mu\phi *)(\partial^u\phi - ieA^\mu\phi) - m^2\phi * \phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (4.19)$$

Let's apply the local U (1) phase transformation:

$\phi^* \to \phi^{*'} = e^{-i\alpha(x,t)}\phi^*$

$D_\mu\phi \to D'_\mu\phi'(\partial_\mu - ieA_\mu)e^{-i\alpha(x,t)}\phi$. We can then use the product rule to obtain:

$$D_\mu\phi = (\partial_\mu - ieA_\mu)e^{-i\alpha(x,t)}\phi = e^{i\alpha(x,t)}\left(\partial_\mu + i(\partial_\mu\alpha)\right)\phi - ieA_\mu e^{i\alpha}\phi = e^{i\alpha(x,t)}(\partial_\mu\phi + i(\partial_\mu\alpha)\phi) - ieA_\mu\phi = e^{i\alpha(x,t)}\left(\partial_\mu - ie\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)\right)\phi.$$

To make sure the covariant derivative is invariant, the gauge field A_μ must transform in the way obtained in the double parentheses. $A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha$

This makes sure that $D_\mu\phi \to D'_\mu\phi' = e^{i\alpha}D_\mu\phi$. So now we can combine the two terms and get:

$$(D_\mu\phi^*)(D_\mu\phi) \to (D'_\mu\phi'^*)(D'_\mu\phi') = e^{i\alpha}(D_\mu\phi^*)e^{-i\alpha}(D_\mu\phi) = (D_\mu\phi^*)(D_\mu\phi).\ \text{Since}\ e^{i\alpha}\cdot e^{-i\alpha} = 1 \qquad (4.20)$$

The same applies to the mass term.

$$m^2\phi^*\phi \to m^2(e^{-i\alpha}\phi^*)(e^{i\alpha}\phi) = m^2\phi^*\phi$$

What about $F_{\mu\nu}$? $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Remember the transformation we obtained to make the covariant derivative invariant:

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha \Rightarrow F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu\left(A_\nu + \frac{1}{e}\partial_\mu\alpha\right) - \partial_\nu\left(A_\mu + \frac{1}{e}\partial_\mu\alpha\right). \qquad (4.21)$$

Since $\partial_\mu\partial_\nu\alpha = \partial_\nu\partial_\mu\alpha \Longrightarrow F'_{\mu\nu} = F_{\mu\nu}$. and every term of the scalar QED is invariant, the whole scalar QED is invariant under the local U (1) phase transformation. Let's now get Noether's current for this. The transformations are

$$\phi \to \phi' = e^{i\alpha(x,t)}\phi,\ \phi^* \to \phi^{*'} = e^{-i\alpha(x,t)}\phi^*,\ A_\mu \to A'_\mu = A_\mu - \frac{1}{e}e\partial_\mu\alpha \qquad (4.22)$$

The variation in the Lagrangian is:

$$\delta L = \frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial\phi^*}\delta\phi^* + \frac{\partial L}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial L}{\partial(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*) + \frac{\partial L}{\partial A_\nu}\delta A_\nu + \frac{\partial L}{\partial(\partial_\mu A_\nu)}\delta(\partial_\mu A_\nu) \qquad (4.23)$$

Integrating the derivative-variation terms by parts at the density level and collecting total derivatives, this separates the pieces that are proportional to the equations of motion from a total divergence:

$$\delta L = \left(\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)}\right)\delta\phi + \left(\frac{\partial L}{\partial\phi^*} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi^*)}\right)\delta\phi^* + \left(\frac{\partial L}{\partial A_\nu} + \partial_\mu F^{\mu\nu}\right)\delta A_\nu + \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial L}{\partial(\partial_\mu\phi^*)}\delta\phi^* - F^{\mu\nu}\delta A_\nu\right] \qquad (4.24)$$

Now to specialize the variations to the infinitesimal local U(1) phase:

$$\delta\phi = i\alpha(x)\phi,\ \delta\phi^* = -i\alpha(x)\phi^*,\ \ \delta A_\nu = \frac{1}{e}\partial_\nu\alpha(x). \qquad (4.25)$$

(For small α we use $e^{\pm i\alpha} \approx i\alpha$.)

Because the Lagrangian is gauge invariant, $\delta S = 0$ for these variations. Next, we take the global subset of the symmetry by setting α constant, so $\delta A_\upsilon = 0$. Substituting (4.24) into (4.23) and using $\delta A_\upsilon = 0$. gives:

$$0 = \left(\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)}\right)i\alpha\phi + \left(\frac{\partial L}{\partial\phi^*} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi^*)}\right)(-i\alpha\phi^*) + \partial_\mu\left[i\alpha\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\phi - \frac{\partial L}{\partial(\partial_\mu\phi^*)}\phi^*\right)\right] \qquad (4.26)$$

The two parenthetical factors multiplying $i\alpha\Phi$ and $(-i\alpha\Phi^*)$ are exactly the left-hand sides of the Euler-Lagrange equations for Φ and Φ^*. When the fields satisfy their equations of motion, those factors will vanish. Dropping that term and removing the overall constant α leads to the local continuity equation:

$$\partial_\mu\left(i\phi\frac{\partial L}{\partial(\partial_\mu\phi)} - i\phi^*\frac{\partial L}{\partial(\partial_\mu\phi^*)}\right) = 0 \qquad (4.27)$$

Therefore, the conserved Noether current is:

$$J^\mu = i\phi\frac{\partial L}{\partial(\partial_\mu\phi)} - i\phi^*\frac{\partial L}{\partial(\partial_\mu\phi^*)}, \qquad \partial_\mu J^\mu = 0. \qquad (4.28)$$

Finally, using the known expressions for the derivative terms in scalar QED,

$$\frac{\partial L}{\partial(\partial_\mu\phi)} = (D^\mu\phi)^*, \qquad \frac{\partial L}{\partial(\partial_\mu\phi^*)} = (D^\mu\phi) \qquad (4.29)$$

To write the current in the familiar form,

$$J^\mu = i(\phi(D^\mu\phi)^* - \phi^*D^\mu\phi), \qquad \partial_\mu J^\mu = 0 \qquad (4.30)$$

Discussion

Noether's theorem is one of the most useful tools for theoretical physics. It has applications wherever there is continuous symmetry in any physical system. Whether the system is local or isolated, the conservation laws that are derived from the symmetry and valid exactly and can be easily applied to simplify problems in both classical and quantum physics.

The importance of Noether's theorem extends even further when we consider different frames of reference. The laws of physics must be invariant in different frames. This requires the introduction of extra structure to maintain that invariance. For example, in non-inertial frames, fictitious forces like centrifugal or Coriolis forces are introduced to make sure that Newton's

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laws are valid in these systems. Similarly, in particle physics, gauge fields are introduced to maintain invariance under local symmetries. Specifically, invariance under local phase shifts in the quantum field of the electron involves introducing the electromagnetic field, which naturally couples to the electric charge.

This principle of extra fields arising to maintain local symmetries gives us valuable insight into reality. They hint at the existence of fundamental interactions like electromagnetism, and the reason there is conservation of electric charge and the existence of light. This principle is also at the bedrock of particle physics. Quarks within protons and neutrons follow a symmetry based on the number three, while discrete symmetries such as charge conjugation (C), parity (P), and time reversal (T) give us valuable insight into understanding particles and anti-particles.

We must, however, be careful not to overextend the theorem into areas it can't. To point to its biggest limitation, the theorem breaks down when there are only discrete symmetries or no symmetry at all. An example of this is where spacetime itself is dynamical. In that case, the underlying symmetries don't hold globally. We see this with the expansion of the universe, which breaks perfect time-translation symmetry. As a result, energy isn't conserved at the cosmic level. This is evidence when we look at the redshift of light, where photons lose energy as their wavelengths stretch with the expanding universe. Similarly, the universe also doesn't have perfect spatial symmetry because of the unequal distribution of stars, planets, and other structures. This implies that we can't apply conservation of momentum globally.

Research around this subject is constantly being done to find out if there are more fundamental symmetries. One popular domain of research is Supersymmetry, which suggests that there might be a deep symmetry between matter particles and force-carrying particles, pointing to a unified framework for the forces of our universe. Whether or not these symmetries hold in nature is still under research, but symmetries and Noether's theorem are at the forefront of shaping modern physics.

Conclusion

As we have shown, Noether's theorem is applicable across many systems in physics. Even in cases such as the gauge symmetry, where there is a redundancy of information, the conserved quantity had to be necessarily true for the Lagrangian to be invariant.

There were many limitations to this paper. As mentioned in the methods, there was a limitation in deriving the Lagrangian of the complex systems. Moreover, many specific cases and extensions of Noether's theorem have not been considered. Noether's theorem assumes that any symmetry under consideration must be continuous. Even though no conserved quantity is derived from such discrete symmetries, they often impose selection rules in quantum systems, which limit possible transitions or interactions. There are many examples, such as Parity Symmetry, Time Reversal Symmetry, and Charge Conjugation Symmetry.

While Noether's theorem and its applications have been well established, there is still much ongoing research concerning its implications and reach. As discussed, there is still ongoing research about spontaneous symmetry breaking. Research continues into how symmetries can be spontaneously broken in various physical systems, which leads to phenomena such as the Higgs mechanism. There can also be extensions in the formalism of discrete symmetries, which do not lead to conserved quantities but can have many physical implications. Of course, there are important considerations that need to be made, such as to what extent Noether's theorem can be applied outside of physics, but that is a question for further research.

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