

Special Functions: Gamma and Beta Functions - Theory and Applications

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ABSTRACT: This article investigates the Gamma and Beta functions, two key special functions in mathematics and mathematical physics. It begins by presenting their integral definitions, basic properties, and their application, including the connection between the Gamma function and factorials and the relationship between the Gamma and Beta functions. The applications are also considered, especially in physics and applied mathematics, where these functions appear in problems involving the ideal classical gas, the harmonic oscillator, probability distributions, and integration formulas. This article highlights the importance of the Gamma and Beta functions as useful tools for solving mathematical problems in science and engineering.

KEYWORDS: Mathematical Physics, Gamma Function, Beta Function, Ideal Classical Gas, Harmonic Oscillator, Integration.

■ Introduction

Mathematics is a fundamental study that supports many fields, including physics, chemistry, natural science, engineering, economics, and architecture. For example, it provides tools for analyzing patterns, modeling real-world phenomena by analyzing statistically, and predicting future events. Although mathematics may seem separate from everyday life, the base of modern society is based on mathematical ideas.

A function is one of the key concepts in mathematics, and some advanced functions appear frequently in mathematics, physics, and engineering, called special functions. For example, the gamma function, beta function, Bessel functions, Legendre polynomials, and spherical functions are special functions. Special functions help solve differential equations, evaluate difficult integrals, and model complex systems from their properties. For instance, the recurrence relation of the Gamma function allows factorials with a continuous number, which can be used to model the time until a specific number of events occur.

This article focuses on the gamma function and the beta function. The gamma function, written as $\Gamma(z)$, extends the idea of factorials beyond positive integers. It is often used in calculus, probability, statistics, physics, and engineering. For example, it is used in the gamma distribution, chi-squared distribution, improper integrals, differential equations, and formulas involving higher-dimensional geometry.¹

The beta function is another important special function defined by an integral. It is closely related to the gamma function and has the symmetry property $B(x, y) = B(y, x)$. It is used in probability theory, especially in defining the beta distribution, and it also appears in physics and engineering problems involving trigonometric and weighted integrals.

Together, the gamma and beta functions allow many complicated expressions and integrals to be written in simpler closed forms. They are widely used in theoretical mathematics, physics, statistics, and scientific computation. This article will introduce their definitions, basic properties, important identities, and applications, including Euler's reflection formula and

examples from quantum physics, statistical mechanics, and integral evaluation.

■ Results and Discussion

Properties of the Gamma Function:

Definition:

The Gamma function is denoted as $\Gamma(z)$, and this is a mathematical function that generalizes the factorial function to complex and real number arguments (except for non-positive integers, where it is undefined). It is defined as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

for complex numbers z with a positive real part.

Recurrence relation:

The recurrence relation shows the value of the function at a given number using the value at the previous number. This relation shows that the value of the gamma function at $z + 1$ is equal to z multiplied by the value of the gamma function at z , which is related to how factorials are related to each other.^{3,4}

When $z > 0$

$$\Gamma(z) = (z - 1)\Gamma(z - 1) \quad (1)$$

The Gamma function at z equals $z - 1$ times the Gamma function at $z - 1$.

Proof:

$$\text{Solve } \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

$$= [x^{z-1}(-e^{-x})]_0^{\infty} + \int_0^{\infty} (z-1)x^{z-2}e^{-x} dx$$

$$\text{Evaluating } [x^{z-1}(-e^{-x})]_0^{\infty} = 0$$

$$x^{z-1}(-e^{-x}) = -\frac{x^{z-1}}{e^x}$$

In this expression, we assume that $z > 1$; therefore, the entire expression approaches zero.

Now we are left with the following:

$$\begin{aligned}\Gamma(z+1) &= [-x^z e^{-x}]_0^\infty + z \int_0^\infty x^{z-1} e^{-x} dx \\ &= z \int_0^\infty x^{z-1} e^{-x} dx \\ &= z\Gamma(z)\end{aligned}$$

From these proofs, we conclude that:

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

Relation to factorial:

The Gamma function generalizes the factorial function to all complex numbers except for negative integers. For every positive whole number n , the value of the gamma function at n is equal to the factorial value of $n - 1$.

$$\Gamma(n) = \Gamma(n-1)!$$

The Gamma function at n is equal to the factorial $n - 1$.^{3,4}

Proof:

The recurrence relation of the Gamma function shows why the Gamma function is connected to the factorial function. Since equation (1),

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

has already been proved, we can use it to show that the Gamma function gives the same values as factorials when the input is a positive integer. First, we check the base case:

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1$$

Also,

$$0! = 1$$

Therefore,

$$\Gamma(1) = 0!$$

Now, using the recurrence relation, we obtain

$$\Gamma(z) = (z-1)(z-2)(z-3)\dots\Gamma(1)$$

Since $\Gamma(1) = 1$, we obtain

$$\Gamma(z) = (z-1)!$$

From these proofs, we conclude that:

$$\Gamma(n) = \Gamma(n-1)!$$

This means that the Gamma function gives the same values as factorials for positive integer inputs, while also allowing similar calculations for non-integer and complex values.³

This can be used in probability and statistics. For example, the Gamma function is used in the gamma distribution.³ This distribution is commonly used to model waiting times like the time until a machine fails or the time between random events.⁵ In these situations, the shape of the distribution does not always have to be a whole number. It may be a decimal value, such as 2.5. By using this factorial relation of the gamma function, it is possible to calculate the decimal value of a factorial.

Special value:

Some special values of the Gamma function are important, allowing us to connect with π . The Gamma function at $1/2$ is equal to $\sqrt{\pi}$.

$$\Gamma(1/2) = \pi$$

In this proof, we use the Gaussian integral, I , which is defined as

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Since this integral is difficult to evaluate directly, we square it:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right)$$

This variable y is only used to create a copy of I . Since they are both integrals of the same product, their product is still I^2 .

I^2 can also be written as:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

To simplify, we use polar coordinates:

$$x^2 + y^2 = r^2$$

Therefore,

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta$$

This integral can be separated:

$$I^2 = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^\infty r e^{-r^2} dr \right)$$

These two integrals can be calculated as:

$$\int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_0^\infty r e^{-r^2} dr = \frac{1}{2}$$

Therefore,

$$I^2 = 2\pi \times \frac{1}{2} = \pi$$

Since $I > 0$,

$$I = \sqrt{\pi}$$

From these proofs, we conclude that:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

This special value of the Gamma function gives exact values, connecting the Gamma function to π . This value is also used to calculate other half-integer values.

For example,

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

They allow us to simplify difficult expressions instead of leaving an answer in complicated form.

Euler's Reflection Formula:

Euler's reflection formula is an identity that relates the Gamma function at z and $1-z$.^{3,4} It is especially important in complex analysis and the theory of special functions.⁴

For complex numbers z where z is not an integer:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Proof:

We begin with Euler's infinite product representation:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right) e^{-\frac{z}{n}}$$

where γ is the Euler-Mascheroni constant.

Now replace z by $-z$. We obtain

$$\frac{1}{\Gamma(-z)} = (-z)e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)e^{\frac{z}{n}}$$

Next, we multiply these two expressions:

$$\frac{1}{\Gamma(z)\Gamma(-z)} = \left[ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)e^{-\frac{z}{n}}\right] \left[(-z)e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)e^{\frac{z}{n}}\right]$$

The exponential factors cancel:

$$e^{\gamma z} e^{-\gamma z} = 1$$

$$e^{-\frac{z}{n}} e^{\frac{z}{n}} = 1$$

Therefore,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)\left(1 - \frac{z}{n}\right)$$

Since $(1+a)(1-a)=1-a^2$,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad (2)$$

Now we use the infinite product expansion of the sine function:

$$\sin \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Therefore,

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \sin(\pi z)}{\pi z}$$

We substitute this into (2),

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \times \frac{\sin \sin(\pi z)}{\pi z}$$

We cancel out z ,

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -\frac{z \sin \sin(\pi z)}{\pi} \quad (3)$$

Now we use the recurrence relation (1),

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

Replace z by $1-z$,

$$\Gamma(1-z) = (-z)\Gamma(-z)$$

Multiply both sides by $\Gamma(z)$,

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z)$$

Substitute for (3),

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \sin(\pi z)}$$

Then,

$$\Gamma(z)\Gamma(1-z) = -z \left(-\frac{\pi}{z \sin \sin(\pi z)} \right)$$

Therefore,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \sin(\pi z)}$$

From these proofs, we conclude that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \sin(\pi z)}$$

Euler's reflection formula gives a relationship between $\Gamma(z)$ and $\Gamma(1-z)$. This allows difficult Gamma function values to be expressed in terms of trigonometric functions. Special values such as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and for the Gamma function at negative non-integer values can be calculated using Euler's reflection formula.

Applications of the Gamma Function:

Quantum Mechanics: Quantum Harmonic Oscillator:

The harmonic Oscillator is one of the important models of physics. Any mass experiencing a restoring force near a stable equilibrium behaves like a harmonic oscillator for small vibrations, and because of this, harmonic oscillators appear throughout nature and form the basis of many human-made technologies, including clocks and radio circuits. In quantum physics, energy comes in fixed, discrete levels rather than a smooth range in the system of a harmonic oscillator.⁶ This simple model turns out to be very powerful because it helps describe vibrations of atoms, molecules, and even the behavior of light in a cavity. We will study in this section how the Gamma function appears in the quantum harmonic oscillator.

A Gaussian moment is the expected value of the power of a variable in a Gaussian distribution. The Gaussian distribution (also called the normal distribution) is a type of symmetric bell-shaped probability distribution. In one-dimensional quantum harmonic oscillators, the wave function of the ground state is Gaussian, so these moments naturally appear.⁶ Therefore, quantities such as $\langle x^{2k} \rangle$ can be directly reduced to Gaussian moments.⁷

Consider the general Gaussian moment:

$$I_n = \int_0^{\infty} x^n e^{-x^2} dx, \quad (n > 1)$$

Let $u=x^2$, $du=2xdx$,

$$x^2 dx = \frac{1}{2} u^{\frac{n-1}{2}-1} e^{-u} du$$

Substitute:

$$I_n = \frac{1}{2} \int_0^{\infty} u^{\frac{n-1}{2}-1} e^{-u} du$$

From the definition of the Gamma function,

$$I_n = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right)$$

Therefore,

$$\int_0^{\infty} x^n e^{-x^2} dx = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right)$$

By symmetry,

$$\int_{-\infty}^{\infty} x^n e^{-x^2} dx = \begin{cases} 0, & n \text{ odd} \\ \Gamma\left(\frac{n+1}{2}\right), & n \text{ even} \end{cases}$$

Example 1: Normalization of the ground-state wave function:

In quantum mechanics, a wave function must be normalized, meaning that the total probability of finding the particle must be equal to 1. Therefore, the ground-state wave function must satisfy

$$\int_{-\infty}^{\infty} |\Psi_0(x)|^2 dx = 1$$

We substitute

$$\Psi_0(x) = A e^{-\frac{\alpha x^2}{2}}$$

obtaining

$$|\Psi_0(x)|^2 = A^2 e^{-\alpha x^2}$$

Therefore, the normalization condition becomes

$$A^2 \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = 1$$

The integral $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx$ is a Gaussian integral. Using the Gamma function, it can be simplified as:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

Therefore,

$$A^2 \sqrt{\frac{\pi}{\alpha}} = 1$$

Solving for A, we get

$$A = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}}$$

Therefore, the normalized ground state wave function is

$$\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha x^2}{2}}$$

This shows that the Gamma function is useful when determining the correct normalization constant for the quantum harmonic oscillator.

Example 2: Expectation value of x in the ground state:

The expectation value of position is defined by

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi_0(x)|^2 dx$$

Substituting the normalized probability density gives

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} x e^{-\alpha x^2} dx$$

The function $x e^{-\alpha x^2}$ is an odd function since replacing x by -x changes its sign. Since the integral is taken over the interval $(-\infty, \infty)$, the positive and negative contributions cancel each other.

Therefore,

$$\langle x \rangle = 0$$

This shows that the average position of the particle is zero in the ground state, meaning that the particle has no preferred direction from the center of the oscillator.

Ideal Classical Gas and the Equipartition Theorem

An ideal gas is a model for a gas with a few assumptions, like negligible particle volume, no intermolecular forces, perfectly elastic collisions, and random motion.⁸ This model follows the ideal gas law, $PV=nRT$. It is easy to handle in statistical mechanics and useful when calculating pressure, volume, mole, and temperature. In reality, gas particles do interact, have a measurable volume, and can liquify or solidify at low temperatures and high pressure.

In classical statistical mechanics, the equipartition theorem gives a direct link between a system's temperature and its average energy.⁹ It is often referred to as the law of equipartition. The basic idea is that, at thermal equilibrium, energy is shared evenly among all available forms, so that each degree of freedom carries the same average kinetic energy.

For a gas particle moving in three dimensions, the particle can move in 3 direction of: x, y, and z. Therefore, it has three translational degrees of freedom. Each direction contributes:

$$\frac{1}{2} k_B T$$

So the average kinetic energy of one particle is:

$$3 \times \frac{1}{2} k_B T = \frac{3}{2} k_B T$$

If a gas contains N particles, the kinetic energy will be

$$U = \frac{3}{2} N k_B T$$

This formula shows that the internal energy of a classical ideal gas is proportional to the number of particles and the temperature. As the temperature increases, the internal energy increases.

This result can be calculated using the Gamma function in a more advanced derivation. The average energy of gas particles can be calculated using integrals over momentum in statistical mechanics. These integrals have the form:

$$\int_0^{\infty} x^{n-1} e^{-\alpha x^2} dx$$

This can be written using the Gamma function:

$$\int_0^{\infty} x^{n-1} e^{-\alpha x^2} dx = \frac{1}{2} \alpha^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)$$

The gamma function gives a more advanced way to obtain the same result through statistical mechanics, giving the internal energy formula,

$$U = \frac{3}{2} N k_B T$$

Properties of Beta Function:

Definition:

The Beta function, denoted as $B(x, y)$, is a special function defined for real numbers

$x, y > 0$. It is given by the integral:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

It is also known as the Euler integral of the first kind.

Symmetry Property:

The symmetry property states that the Beta function is unchanged if we swap its two input values:¹⁰

$$B(m, n) = B(n, m)$$

This means the Beta function treats its two arguments equally. Geometrically, this reflects the symmetry of the integrand when the variable t is replaced with $1-t$. For example, $B(2, 3)$ is equal to $B(3, 2)$.

Proof:

We start from the definition

$$B(m, n) = \int_0^1 t^{m-1} (1-t)^{n-1} dt$$

Now make the substitution $t = 1 - u$. Then

$$dt = -du$$

and when t goes from 0 to 1, u goes from 1 to 0.

So the integral becomes

$$B(m, n) = \int_{t=0}^1 t^{m-1}(1-t)^{n-1} dt = \int_{u=1}^0 (1-u)^{m-1}u^{n-1}(-du)$$

Reversing the limits of the integral gives

$$B(m, n) = \int_0^1 u^{n-1}(1-u)^{m-1} du$$

which is the definition of $B(n, m)$, therefore,

$$B(m, n) = B(n, m)$$

From this proof, we conclude that swapping the two inputs of the Beta function does not change its value. This symmetry often simplifies calculations and formulas. For example, in statistics, the Beta distribution has two “shape” parameters, and its formula does not change if those two parameters are swapped. In integrals, this symmetry means you can swap the roles of the two powers and still get the same result.

Relation with Gamma Function:

This property connects the Beta function to the Gamma function, $\Gamma(z)$, which generalizes the factorial. The relationship is:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (4)$$

Recall that for natural numbers:

$$\Gamma(n) = (n-1)!$$

So, the Beta function can also be written using factorials when x and y are whole numbers. This formula is extremely useful in simplifying expressions involving Beta and Gamma functions.³

Proof:

To start with, we will rewrite (4) as:

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x, y)$$

Next, we will look at the Gamma function definitions,

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u} du$$

and

$$\Gamma(y) = \int_0^\infty v^{y-1}e^{-v} dv$$

Multiplying these two integrals will give

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty u^{x-1}v^{y-1}e^{-(u+v)} dudv \quad (5)$$

Simplify by

$$r = u + v$$

and

$$z = \frac{u}{u+v}$$

The Jacobian for this substitution is

$$du dv = r dr dz$$

After substitution, the integral (5) becomes

$$\int_0^1 \int_0^\infty (rz)^{x-1}[r(1-z)]^{y-1}e^{-r} r dr dz$$

Now separate the powers of r , z , and $1-z$

$$(rz)^{x-1}[r(1-z)]^{y-1}r = r^{x+y+1}z^{x-1}(1-z)^{y-1}$$

Therefore, the integral becomes

$$\int_0^1 \int_0^\infty r^{x+y+1}e^{-r}z^{x-1}(1-z)^{y-1} dr dz$$

This can be written as

$$\Gamma(x)\Gamma(y) = \left(\int_0^\infty r^{x+y-1}e^{-r} dr \right) \left(\int_0^1 z^{x-1}(1-z)^{y-1} dz \right)$$

The first part of this integral is the Gamma function:

$$\int_0^\infty r^{x+y-1}e^{-r} dr = \Gamma(x+y)$$

The second part of this integral is the Beta Function:

$$\int_0^1 z^{x-1}(1-z)^{y-1} dz = B(x, y)$$

Therefore,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

which equals

$$B(x, y+1) = \frac{y}{x+y}B(x, y)$$

This relates the beta function to the gamma function. Applying this allows us to calculate beta expressions using given gamma values. For example, it ensures that the sum of all possible outcomes in a beta probability equals 1. In calculus, it helps calculation by simplifying beta integrals into gamma values.

Reduction Formula:

The reduction formula is a useful identity that allows us to express a Beta function with one of its parameters increased by 1 in terms of the original Beta function. The Beta function can be difficult to compute directly for all values. But with this formula, we can reduce a harder case to an easier one. We want to show:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Proof:

Using the relation between the Beta and Gamma functions,

$$B(x, y+1) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y+1)}$$

substitute $y+1$

$$B(x, y+1) = \frac{\Gamma(x)y\Gamma(y)}{(x+y)\Gamma(x+y)}$$

Using the recurrence relation of the Gamma function,

$$\Gamma(y+1) = y\Gamma(y)$$

and

$$\Gamma(x+y+1) = (x+y)\Gamma(x+y)$$

Therefore,

$$B(x, y+1) = \frac{\Gamma(x)y\Gamma(y)}{(x+y)\Gamma(x+y)}$$

This can be written as

$$B(x, y + 1) = \frac{y}{x + y} \times \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

Going back to the relation between the Beta and Gamma functions,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = B(x, y)$$

Therefore,

$$B(x, y + 1) = \frac{y}{x + y} B(x, y)$$

These formulas let us calculate $B(x, y)$ by reducing x or y by 1. This is useful when evaluating integrals of polynomials or trigonometric powers. Each application of the reduction formula gets rid of one power. For example, it is used in integration techniques when reducing $\int \theta \, d\theta$ or $\int \theta \, d\theta$ one power at a time.

Trigonometric Representation:

This formula gives a new way to compute the Beta function, not with powers of t , but with powers of $\sin \theta$ and $\cos \theta$, where θ ranges from 0 to $\pi/2$.¹⁰ We show that

$$B(x, y) = 2 \int_0^{\pi/2} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} d\theta$$

Proof:

We begin with the definition of the Beta function:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad (5)$$

We are going to change the variable of integration from t to θ . Let,

$$t = \theta$$

First, we find how the limit changes from this substitution:

$$t = 0 \Rightarrow \theta = 0 \Rightarrow \theta = 0$$

$$t = 1 \Rightarrow \theta = 1 \Rightarrow \theta = \pi/2$$

Now, differentiate:

$$t = \theta$$

$$dt = 2 \sin \theta \cos \theta \, d\theta$$

Now we substitute into the original integral (5):

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

Since

$$t^{x+1} = (\theta)^{x-1} = (\sin \theta)^{2x-2}$$

and similarly,

$$(1-t)^{y-1} = (\theta)^{y-1} = (\cos \theta)^{2y-2}$$

Therefore

$$B(x, y) = \int_0^{\pi/2} (\sin \theta)^{2x-2} (\cos \theta)^{2y-2} 2 \sin \theta \cos \theta \, d\theta$$

Now, we can simplify by combining powers:

$$(\sin \theta)^{2x-2} \times \sin \theta = (\sin \theta)^{2x-1}$$

$$(\cos \theta)^{2y-2} \times \cos \theta = (\cos \theta)^{2y-1}$$

Therefore,

$$B(x, y) = 2 \int_0^{\pi/2} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} d\theta$$

This is the trigonometric representation of the Beta function. It is especially useful for integrals that involve powers of sine and cosine. Such integrals appear when finding areas under curves or volumes of solids with circular symmetry. In physics and engineering, this representation also appears when studying sinusoidal signals, such as sound waves, alternating electric currents, and vibrations. The Beta function helps turn these complicated trigonometric integrals into simpler expressions.

Conclusion

This article has summarised the key properties of the Gamma function and Beta function. It covers the definition of the Gamma function to start with, then proves the recurrence relation, which shows that the value of the Gamma function at $z + 1$ is equal to z multiplied by the value of the Gamma function at z .³ From this relation, we covered the factorial relation. We also discussed the special values, including the calculation of $\Gamma(1/2) = \sqrt{\pi}$. In addition, we proved Euler's Reflection Formula, which is an identity that relates $\Gamma(z)$ and $\Gamma(1 - z)$ through the sine function.

For the Beta function, we showed its symmetry property. This means that the order of the arguments does not matter. We then proved its close relationship with the Gamma function. This connection highlights how the Beta function can be expressed using Gamma functions, which can be applied to simplify calculations and solve more complex problems. We also explored the reduction formula, which allows more complicated Beta function expressions to be simplified. The trigonometric representation, which ties the Beta function to integrals involving sine and cosine, was also proved. Finally, we covered special cases such as $B(1, n)$, which simplify simple fractions.

We also introduced Gaussian moments and the one-dimensional quantum harmonic oscillator to show how these functions are applied in physics, particularly for describing distributions and quantum systems. For example, the Gamma function helps calculate probabilities and energies in these contexts. In addition, we discussed the classical ideal gas in three dimensions and its internal energy. We have explained how these mathematical tools can be applied to real physical systems in thermodynamics. Also, the ideal classical gas, where we looked at the internal energy of a classical ideal gas in 3 dimensions, can be applied to chemistry, physics, and other various fields.

In conclusion, these properties allow us to calculate more complicated problems more simply. Because of their extremely useful properties, they are applied in various fields such as physics, probability and statistics, calculus, engineering, and many others. The Gamma function and Beta function described in this manuscript are widely used in the real world.

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