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Towards a New Hybrid Stochastic Model for Enhanced Stock Price Estimation: The Heston-GARCH-Levy Model

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ABSTRACT: The objective of this paper is to look at a stochastic framework for stock price estimation that builds upon specific advanced structures used in the financial industry. Specifically, we look at and simulate the Heston Model (with correlated Wiener processes with correlation coefficients), and other variants that are considered more accurate with respect to Black-Scholes. Then we look at an improved version of this model with jump conditions and simulate it, then look at the Stochastic Alpha Beta Rho (SABR) model and the Rough Volatility Model. We finally construct a possible Heston-GARCH-Levy Model with Jump Diffusions, which has a Heston Stochastic Volatility, a GARCH for Conditional Heteroskedasticity, as well as Jump Diffusions that introduce discontinuities in the price process, with Levy processes for tailed behavior. This combined model is then analyzed as a Hybrid Stochastic System specifying assumptions, the function spaces and norms needed, existence and uniqueness proofs, handling of hybrid components, regularity and stability bounds, counterexamples when solutions fail, and looking at BIBO stability of the system. Our analysis has resulted in a final stochastic model that has passed conditions for uniqueness and existence of solutions.

KEYWORDS: Mathematics, Analysis, Stochastic, Ito Calculus, Heston-Model, Cox-Ingersoll-Ross, GARCH & Levy, Processes for Jump Diffusions.

Introduction

The modeling of financial markets has seen significant advances, from the seminal Black-Scholes model to stochastic volatility frameworks such as the Heston model. However, real-world asset prices exhibit characteristics that traditional models fail to capture, such as volatility clustering and heavy tails. This paper explores the integration of the Heston model with GARCH processes and Levy noise to address these limitations. The combined model leverages:

- Stochastic volatility dynamics for market fluctuations
- GARCH processes to account for volatility clustering
- Levy noise to model jumps and heavy-tailed return distributions

Background and Past Innovations

1. The Black-Scholes Model: A Starting Point:

The Black-Scholes model, made in 1973, was groundbreaking in financial mathematics, providing a closed-form solution for options pricing.[1] The stock price S(t) is modeled as a geometric Brownian motion:

$$dS_t = S_t\,(rdt + \sigma dW_t)$$

Here, r is the risk-free interest rate, the Greek letter sigma represents the constant volatility, and W(t) is a standard Brownian motion. The Black-Scholes model assumes constant volatility and a lognormal price distribution, making it analytically tractable but inconsistent with real market behavior.[1] Price distributions in real-world asset returns are often skewed, and thus, a normal assumption is inaccurate for accurate price modeling.

2. Limitations of the Black-Scholes Model:

Empirical studies have highlighted major shortcomings in the Black-Scholes framework.

- Volatility smiles and skews: Real option prices show implied volatilities vary with strike price and maturity, contradicting the constant volatility assumption
- Fat Tails in Return Distributions: Empirical stock returns exhibit heavy tails and leptokurtic behavior, unlike the normal distribution predicted by Black-Scholes.
- Sudden price jumps: Financial markets experience large discontinuous moves like crashes and new shocks, which a pure diffusion process fails to capture.
- Volatility clustering: Periods of high volatility tend to be followed by more volatility, a missing feature in the constant σ assumption.

Below is a sample simulation of the Black-Scholes model tested against Apple Stock prices.

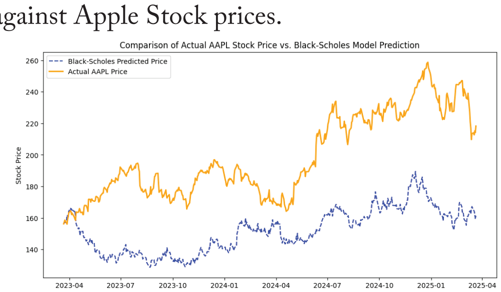

Figure 1: Predicted AAPL prices against Black Scholes. Black-Scholes model simulation was tested against AAPL stock prices for the last 2 years, showing extreme errors and only a few solutions at the beginning of the simulation. Simulation was done with the Monte-Carlo simulation method, and significant gaps and high residuals are present between the two simulations.

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Notice the significant gaps between the predicted and actual prices. 1's comparisons have nearly no solutions, with only incredibly small solutions (where intersections occur) happening at the beginning. Thus, while the figure demonstrates that the Black-Scholes model can predict in the short run, it is eventually ineffective over a span of weeks. The residuals are very large, and thus the model has weak predictions due to the reasons mentioned previously. It can, however, be argued that this is simply due to computational errors. The simulation is utilizing incredible amounts of approximation methods to predict stock prices in the following period. It is analogous to Euler's method or linear approximation for ordinary calculus, using the tangent line and derivative at step size increments to predict future values of solutions to differential equations. As we span out in time, approximations capture more inaccuracies. However, as we will show in the results section, this error does not inhibit the Heston-GARCH-Levy to the extent of the Black Scholes errors, meaning the HGL model is indeed more mathematically effective.

Methods

The Heston model here considers stochastic volatility, adding a dynamic stochastic evolution of variance with a second differential equation.[2] The continuous Heston variance evolution follows a mean-reverting Cox-Ingersoll Ross process.[3]

$$dv_t = \mathcal{K}(\theta - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\, dW_t^v$$

This allows volatility to fluctuate randomly over time, improving the model's fit to market data. Here, $\mathcal{K}$ is the mean-reverting rate, θ is the long-term memory, and the product of the square root volatility, and σ gives us the volatility of volatility. However, the Heston model still assumes continuous price paths, failing to capture market jumps and fat tails.

Thus, to capture market irregularities, we incorporate a Levy jump process following a Poisson process, leading to our Stochastic Differential Equation (SDE) for the dynamics of the stock price as:

$$dS_t = S_t\,(rdt + \sigma dW_t + J_t dN_t)$$

The additional J(t) term is this jump process following a Poisson distribution N(t) with intensity denoted by the Greek letter lambda. J(t) are specifically i.i.d. jump sizes. Levy processes further generalize jumps by allowing infinite activity jumps, such as the Variance Gamma and Normal Inverse Gaussian, capturing fat tails and skewness in stock returns more effectively than Merton's model.[4]

Lastly, we incorporate a GARCH component into our model. To better model long-memory effects in volatility, we incorporate a GARCH (Generalized Autoregressive Conditional Heteroskedasticity) process that innovates upon the general ARCH(1,1) model for volatility clustering.[5] The GARCH model is defined as follows:

$$v_{t+1} = w + \alpha\epsilon_t^2 + \beta v_t\,,\ \epsilon_t = \sqrt{v_t}Z_t$$

Unlike the Heston model, GARCH models account for discrete-time volatility clustering, making them effective in high-frequency financial modeling. Notice its recursive behavior, as there is a v_{t+1} that is dependent on the previous period ϵ_t to predict future volatility fluctuations. Therefore, the Hybrid-GARCH-Levy Model with Jump Diffusions integrates:

- Heston's Stochastic volatility (continuous reverting behavior)
- GARCH dynamics (discrete volatility clustering)
- Levy-driven jumps (heavy tails and skewness)
- Poisson processes (sudden price jumps)

Advantages over previous models include capturing volatility clustering (GARCH and Heston components ensure time-varying volatility), modeling extreme market moves (Levy jumps introduce fat tails and rare events), flexible skewness and kurtosis (the model accommodates asymmetric return distributions), and better option pricing fits (the combination of stochastic volatility and jumps corrects the implied volatility smile). The Heston-GARCH-Levy model with Jump Diffusions represents an improvement over classical models, allowing for realistic asset price dynamics. By addressing volatility clustering, jumps, and heavy tails, it better explains market phenomena such as crashes, skewed option prices, and persistent volatility shocks. The model is also a hybrid model, meaning that there are continuous and discrete dynamics occurring simultaneously. We will prove such a structure exists and contains unique solutions in the following section. And so, we propose a new system of SDEs to capture real-world market dynamics, for which we will prove in the following section:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t + S_t J_t dN_t\ ,$$

$$dv_t = \mathcal{K}(\theta - v_t)dt + \sigma\sqrt{v_t}\, dW_t^v$$

$v_t = \alpha + \beta v_{t-1} + \gamma\epsilon_{t-1}^2$ (Note: The v(t+1) was simply rewritten as v(t), moving the starting point of the recursion sequence back by 1).

- S(t) is the stock price at time t,
- v(t) is the stochastic variance process,
- μ is the drift of the stock price,
- $\mathcal{K}$ is the mean-reversion rate of variance
- σ is the volatility of variance
- W(t) and Z(t) are correlated Brownian motions with a correlation coefficient ρ
- J(t) models the jump sizes,
- N(t) is a Poisson process modeling the jump occurrences,
- λ is the intensity of the Poisson process.

Theorems for Stability Proofs (Methods)

A major component of this paper will be to extend results from Ordinary Differential Equations to Stochastic Differential Equations, under appropriate conditions. The key intuition is that an SDE is an ODE perturbed by noise, often in the form of Brownian motion. In particular, we are concerned with four specific ideas:

- Existence and Uniqueness: How ODE theorems extend to SDEs
- Stability and Convergence: How solutions behave under perturbations
- Gronwall's Inequality for SDEs: A key inequality that carries over from ODEs

• Flow properties and Diffeomorphisms: How ODE flow maps extend to stochastic settings

Consider a deterministic ODE of the form:

$$\frac{dx_t}{dt} = f(x_t), \ \ x_0 = x$$

By Picard's existence and uniqueness theorem, if f(x) is Lipschitz continuous, there exists a unique solution x(t) for all t.[6] Now consider the corresponding stochastic differential equation:

$$dX_t = f(X_t)dt + g(X_t)dW_t$$

Here, W(t) is standard Brownian motion, and g(x(t)) represents the diffusion terms. Here we have the following theorem. If (Lipschitz Condition), $|f(x) - f(y)| + |g(x) - g(y)| \leq C|x-y|$ and (Linear Growth Condition), there exists a C>0 such that $|f(x)|^2 + |g(x)|^2 \leq C(1+|x|)^2 \ \forall x$ then there exists a strong unique solution X(t) to the SDE.

In terms of proving this, there is a form of Picard Iteration for ODEs and for SDEs (which is simply an extension). For ODEs, we define a sequence as follows:

$$x_{n+1}(t) = x + \int_0^t f(x_n(s))ds$$

This converges under the Lipschitz condition. In the case of SDEs, we extend the above to SDEs:

$$x_{n+1}(t) = x + \int_0^t f(x_n(s))ds + \int_0^t g(X_n(s))dW_s$$

The deterministic part follows from Picard iteration for ODEs. The stochastic integral exists due to Ito's Isometry. In the case of proving convergence, we use Banach's fixed-point theorem in the space of stochastic processes.[7]

$$\mathrm{E}\sup_{0\leq t\leq T} |X_{n+1}(t) - X_n(t)|^2 \to 0$$

This guarantees that the stochastic sequence converges, proving existence and uniqueness with the difference in iterated guesses being zero.[8] Note, for this paper, the capital letter "E" will denote taking the expectation or mean of some process.

Next, let us consider stability and convergence for SDEs. Consider a deterministic stability case with Lyapunov's method for ODEs. For the ODE:

$dx(t)/dt = f(x(t))$

$$\frac{dV}{dt} \leq -cV(x)$$

for some c>0, then x(t) is globally stable and converges to an equilibrium.

Now consider stochastic stability (Ito's Lemma for SDEs). Suppose we have the SDE:

$$dX_t = f(X_t)dt + g(X_t)dW_t$$

Using Ito's Lemma, the stochastic analog of Lyapunov's condition is:

$$\mathcal{L}V(x) = f(x)V'(x) + 1/2\, g^2(x)V''(x)$$

If we can show that $\mathcal{L}V(x) \leq -cV(x)$ then X(t) is stochastically stable. A bit of background is needed here. In the context of SDEs, the operator $\mathcal{L}$ is known as the infinitesimal generator of stochastic processes. It is the key idea in stochastic analysis and helps in deriving stability conditions and solving PDEs associated with SDEs.[9] Two equations in this regard would be the well-known Fokker-Planck equation and the Hamilton-Jacobi-Bellman (HJB) equation, through which the stochastic differential equations can be re-formed into a solvable partial differential equation.[10]

Gronwall's inequality for SDEs is well known, as it is for ODEs. Given:

$$y(t) \leq C + \int_0^t ky(s)ds$$

then $y(t) \leq Ce^{kt}$

The stochastic version is extremely similar. Suppose we have the SDE

$$dX_t = f(X_t)dt + g(X_t)dW_t$$

We apply Ito's Lemma to the process Y_t = |X_t|^2, which gives $dY_t = CY_t dt + g^2(X_t)dt$

This gives

$$Y_t \leq C + \int_0^t kY_s ds$$

So Gronwall's inequality in this case is

$\mathrm{E}[X_t]^2 \leq Ce^{kt}$

This is crucial for bounding solutions for ODEs to ensure solutions do not explode.

Proofs

In order to look at the existence and uniqueness of solutions for this type of Hybrid model, where there are discrete and continuous systems of equations at play, we need to understand the mathematical frameworks underlying its components. Understand that the model combines the following aspects: SDEs from the Heston model volatility, GARCH dynamics, which are discrete-time processes modeling conditional variance and volatility clustering, Levy processes, for tailed behavior, and jump diffusions, which add discontinuities in the price dynamics. We will show well-posedness of our model in this section, show boundedness and uniqueness of solutions, and demonstrate the ways in which we arrive at our final integrated solution of our system of SDEs.

1. Existence and Uniqueness of Solutions:

Let us analyze the variance process for existence and uniqueness, given the framework laid out in Section 4. The stochastic variance process is governed by $dv_t = \mathcal{K}(\theta - v_t)dt + \sigma\sqrt{v_t}Z_t$

The drift term $\mathcal{K}(\theta - v_t)$ and diffusion term $\sigma\sqrt{v_t}$ satisfy the following conditions:

• Local Lipschitz Continuity: The square root volatility function is locally Lipschitz for v(t)>0. This ensures that small changes in v(t) result in small changes in dv(t).

• Linear Growth Condition: The terms $\mathcal{K}(\theta - v_t)$ and $\sigma\sqrt{v_t}$ grow linearly in v(t), satisfying:

$||\mathcal{K}(\theta - v_t)|| + ||\sigma\sqrt{v_t}|| \leq C(1+||v_t||)$ for some constant C>0.

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2. Stock Price Process:

The stock price process includes a jump component:

$$dS_t = S_t\,(\mu d_t + \sqrt{v_t}dW_t + J_t dN_t)$$

The terms in this equation satisfy:

• **Local Lipschitz Continuity:** The three dynamic functions (jump, stock price, and volatility) are Lipschitz in their respective domains, ensuring stability in the evolution of S(t).

• **Linear Growth Condition:** The drift, diffusion, and jump terms grow linearly in, satisfying:

$$||\,\mu S_t\,|| + ||\,\sqrt{v_t}S_t\,|| + ||J_t S_t\,|| \leq C(1 + ||\,S_t)$$

In hindsight, the Lipschitz condition implies that no matter what sequence of events is occurring, and despite the position of the current real-world price, the stock price dynamics cannot change unnaturally. Without such a condition, the model becomes unstable and can predict changes in the price that are nowhere near realistic.

3. Jump Conditions:

The jump component J(t)dN(t) is modeled as a compound Poisson process. The following conditions ensure the existence and uniqueness of solutions:

• The Poisson process N(t) has finite intensity λ over any finite time interval.

• The jump sizes J(t) are modeled to have finite variance, i.e., E[$J_t^{\,2}$] < ∞.

• The jump coefficient J(t) is locally Lipschitz, ensuring the stability of the jump term.

4. The Feller Condition, Simulation Parameters, and GARCH Constraints:

To ensure volatility remains strictly positive, ensuring stability for our model and avoiding negative volatility predictions, we require the Feller condition to be satisfied.[11]

$2K\theta \geq \sigma^2$

This condition above guarantees the model avoids any undefined behavior at [v(t)]^0.5. The boundary classification results for the CIR process are already conventionally discussed in literature, and the Feller condition is sufficient to uphold positive volatility.[12] The uniqueness aspect of SDEs is usually seen with Lipschitz continuity as well as linear growth conditions, and thus validates the use of the Feller condition.

For the GARCH dynamics, which has a discrete time variance, we need non-negativity through the following parameters: $w > 0, \alpha, \beta \geq 0, \alpha + \beta < 1$. If $\alpha + \beta < 1$ then the process is stationary and ergodic, which guarantees a well-defined sequence.

In this paper, the simulation will mainly be based on Monte Carlo simulation methods. The parameters for the simulation of our stochastic dynamics will run 100-500 stochastic paths, then take the median of those paths. We avoid the mean as generally done in Monte Carlo simulations for resistance to outliers and to help the simulation run time. Additionally, the parameters and variables are the same as mentioned in section 3. We constrain volatility from Heston's equations to be strictly positive, and we constrain the GARCH parameters as discussed above.

4.1. Data Collection and Preprocessing::

We download historical stock price data for the desired period. For example, we use Yahoo Finance or other financial data sources to retrieve the adjusted closing prices. We then calculate logarithmic daily returns as r(t) = ln (S_t / S_{t-1}), where S_t is the adjusted closing price at time t. We optimize the model parameters with the Mean Squared Error (MSE) objective function. It is defined as

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{S}_i - S_i\right)^2.$$

We formulate the stock price dynamics under the model

$$S_{t+1} = S_t \exp\left((\mu - 0.5\,v_t)\Delta t + \sqrt{v_t}\Delta t Z_t + \text{Jumps}\right),$$

where we have variance at time t, and Z_t is Brownian Motion. "Jumps" are indeed Poisson-driven.

5.Picard Iteration for Existence and Uniqueness:

In our proof of existence, we use the Picard Iteration Scheme to construct the solution to the stochastic hybrid system. This method is well-suited due to the following reasons:

• It ensures the existence of a unique fixed-point solution under appropriate contraction conditions.

• It aligns naturally with the structure of our SDE.

• It provides a constructive approach to approximating the solution, which is beneficial for both theoretical analysis and numerical implementation.

We consider a stochastic hybrid system of the form:

$$dS_t = f(S_t, v_t, N_t, J_t)dt + g(S_t, v_t, N_t, J_t)dW_t + h(S_t, v_t, N_t, J_t)dN_t$$

where:

• f, g, and h are measurable functions ensuring well-posedness.

• W(t) is a Wiener process.

• N(t) is a Poisson process representing jumps.

• J(t) are i.i.d. jump sizes with finite variance.

Given an initial condition S(0), we seek a solution satisfying:

$$S_t = S_0 + \int_0^t f(S_u, v_u, N_u, J_u)du + \int_0^t g(S_u, v_u, N_u, J_{uj})dW_u + \int_0^t h(S_u, v_u, N_u, J_u)dN_u$$

The Picard Iteration Scheme constructs an approximate sequence $\{S_t^{\,n}\}$ recursively as follows:

$$S_t^{n+1} = S_0 + \int_0^t f(S_u^n, v_u, N_u, J_u)du + \int_0^t g(S_u^m, v_u, N_u, J_u)dW_u + \int_0^t h(S_u, v_u, N_u, J_u)dN_u$$

where $S_t^{\,0} = S_0$ is the initial guess.

To justify why such an iteration is allowed, we will showcase a contraction mapping property.

For a Picard iteration to converge, we need to show that the mapping:

$$T(S)(t) = S_0 + \int_0^t f(S_u^n, v_u, N_u, J_u)du + \int_0^t g(S_u^m, v_u, N_u, J_u)dW_u + \int_0^t h(S_u, v_u, N_u, J_u)dN_u$$

is a contraction in the normed function space $L^2(\Omega; C([0,T];\mathrm{R}))$ Consider two iterates $S_t^{\,n}$ and $S_t^{\,n+1}$ Taking the difference between two iterates and applying Lipschitz conditions gives us the following: $\mathrm{E}\sup_{t\in[0,T]}|S_t^{n+1} - S_t^n|^2 \leq CT\mathrm{E}\sup_{t\in[0,T]}|S_t^n - S_t^{n-1}|^2$. We then force T to be sufficiently small to achieve a contraction as follows: $\mathrm{E}\sup_{t\in[0,T]}|S_t^{n+1} - S_t^n|^2 \leq \theta\mathrm{E}\sup_{t\in[0,T]}|S_t^n - S_t^{n-1}|^2$. For Some 0< θ< 1 And finally, as it is a contraction, by Banach's Fixed-Point Theorem, (S(t))^n converges to a unique solution. Now, to apply such a concept fully to our model, we show that the solution of our system of SDEs, which are denoted as X(t) = (S(t), v(t)) with an initial guess of the zeroth term of the iterated sequence of X(t) for all t on the interval [0, T], converges by estimating the difference between consecutive approximations, and showing that such differences tend towards 0. Above was the framework that we now apply to our H.G.L. model. We define the n-th approximation X_t^{n+1} to be defined recursively as:

$X_t^{n+1} = x_0 + \int_0^t b(s, X_s^{(n)})ds + \int_0^t \sigma(s, X_s^{(n)})dW_s$. We want to show that the difference between consecutive approximations

$$\| X_t^{(n+1)} - X_t^{(n)} \| \leq \int_0^t \| b(s, X_s^{(n)}) - b(s, X_s^{(n-1)}) \| ds + \int_0^t \| \sigma(s, X_s^{(n)}) - \sigma(s, X_s^{(n-1)}) \| dW_s$$

goes to zero. Using Lipschitz conditions, we find an inequality that follows from Itô Isometry:

$\| X_t^{(n+1)} - X_t^{(n)} \| < C \int_0^t \| X_s^{(n)} - X_s^{(n-1)} \| ds$ in which $||X_t^{(n+1)} - X_t^{(n)}|| \to 0$ as $n \to \infty$ uniformly in t $\in [0, T]$ by Gronwall's Inequality.

What this means is that if we take any two guesses of predicted stock prices at the n-th time in the future, the difference between the guesses would be zero, meaning those guesses are forced to be the same.

Thus, there is a unique solution, or no two guesses that differ as the limit goes to infinity in terms of n. And so, we further justified the existence of a unique solution to our stochastic dynamics.

6. Justification for L^2 Normed Space

L^2 provides a reasonable space for our model to work in. Firstly, this normed space has Hilbert Space properties, as L^2 is a Hilbert space, meaning that it has inner products that allow for powerful tools like orthogonality, projections, and energy estimates. We also have Ito Isometry: Stochastic processes, especially those involving Brownian motion, are naturally well behaved in L^2 due to Ito Isometry, which ensures that expectations of squared integrals behave nicely. Well-posedness and many existence and uniqueness results in SDEs and PDEs rely on energy methods in L^2. Below is a demonstration of how other normed spaces fail. Let us look at integral control for the normed space L^(infinity). Such a norm is given by

$$\| f \|_{L^\infty} = \sup_x |f(x)| .$$

A key issue is that the integral of f(x) can diverge, making standard norm-based convergence arguments in L^2 inapplicable.

Counterexample: Consider the sequence of functions g_n of x where it is 1 for x on [0,1] and 1/n for x in (1, n]. In this normed space, all of these functions satisfy $||g_n||_{L^\infty} = 1$. Thus, they do not vanish in the norm as n goes to infinity, which prevents the use of limit arguments that are valid in L^2.

7. Moment Matching for the Hybrid Nature of the Model:

In our system of SDEs, we indeed have a discrete and continuous interplay between the SDEs. The GARCH process and the Levy Jumps are discrete events, which are in place to capture market irregularities that happen in discrete time events. The Heston volatility dynamics are continuous. To ensure stability and the model's accuracy, we need to ensure that the hybrid components match with each other in time. The first consistency condition requires that the moments of discrete-time variance align with the moments of the continuous-time variance process in our system of SDEs. For the Discrete-Time Variance, which is from the GARCH equation, we have: $v_{t+1} = w + \alpha\epsilon_t^2 + \beta v_t$, where $\epsilon_t^2 = \sqrt{v_t}Z_t$ and Z(t) follows a standard normal distribution. We compute the expectation of the GARCH volatility defined by $E[v_{t+1}] = w + \alpha E[\epsilon_t^2] + \beta E[v_t]$. Since $E[\epsilon_t^2] = E[v_t Z_t^2] = v_t$ we have $E[v_{t+1}] = w + \alpha E[v_t] + \beta E[v_t]$. It can be shown that the expectation of the squared GARCH volatility process has a summation in the form of a geometric series, particularly, $\omega \sum_{i=1}^{\infty} (\alpha+\beta)^i$ which converges to $\frac{\omega}{1-(\alpha+\beta)}$ for $\alpha+\beta < 1$ due to the convergence rule for a geometric series. It is why we also set the condition $\alpha+\beta < 1$ for this process. For the Heston variance process, the Fokker-Planck equation implies that the expected variance satisfies: $\frac{dE[v_t]}{dt} = \kappa(\theta - E[v_t])$. Of course, the Feller condition stated earlier guarantees that this volatility process remains strictly positive. To see this visually, the following graph showcases our model's volatility remaining strictly positive over a time interval.

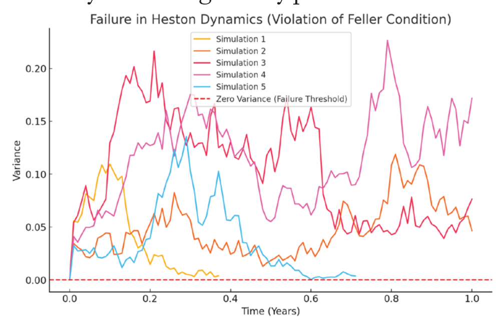

Figure 2: Variance over time of the hybrid model with a threshold for when volatility does not remain stable. The figure showcases 5 different probable expectations of simulations that are above the 0-volatility line, indicating that volatility is strictly positive and the HGL does not blow up.

Figure 2 demonstrates that our volatility dynamics of the Heston aspect of the model will strictly remain positive. Our model is thus steady and will not predict negative volatility, which is both unintuitive and causes model breakdowns. The Feller condition is then satisfied, and we are in a steady state of the model.

In a steady state, $E[v_t] = \theta$. And therefore, for our consistency condition to be met, we require $\theta = \frac{\omega}{1-(\alpha+\beta)}$.

As for the conditions for the jump term J(t), we simply require that the expectation of 1+J(t) is finite and greater than 0 almost surely.

Result and Discussion

We now take our three SDEs in the system of stochastic differential equations for the Heston-GARCH-Levy, and arrive at an equivalent integrated form of the model:

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s\right) \prod_{i=1}^{N_t} (1 + J_{t_i})$$

Here is how the product term arrived in this final form: When a jump occurs in the process S(t) due to the Poisson process, the stock price experiences a discontinuous change. The cumulative effect of jumps on the stock price over the interval [0, T] is described by $\int_0^T J_t dN_t = \sum_{i=1}^{N_T} J_{t_i}$. With this established, and using Monte Carlo simulations, the HGL model was tested against AAPL prices, and the error was analyzed as

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well. Following the conventional work on Heston's model, the variables are defined as follows: the drift function will include" r," which is the risk-free interest rate that is derived using Ito's Formula. We have two separate volatility integrals, one following an ordinary differential, and the other being with respect to Brownian motion. These are, of course, governed by both GARCH and Heston dynamics.

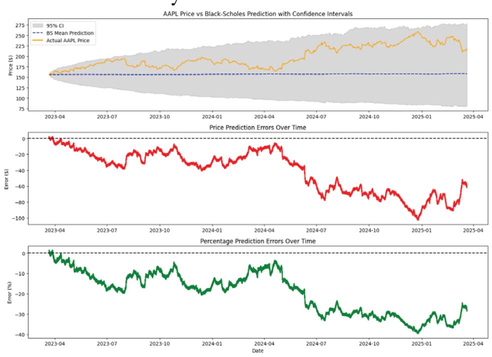

Figure 3: Black Scholes model was tested against AAPL stock prices, and percentage and numerical error charts. A maximum percentage error of 40 percent was shown in this sample simulation.

Figure 3 demonstrates an error analysis simulation of the Black Scholes model when tested against AAPL stock prices. In particular, the error percentages and specific price residuals are displayed over time. The error maximizes at around 40 percent, and the simulation itself has no solutions. The simulation utilized 99 percent confidence intervals, where we are 99 percent confident that the true expected simulation lies within such a range determined by the shaded region.

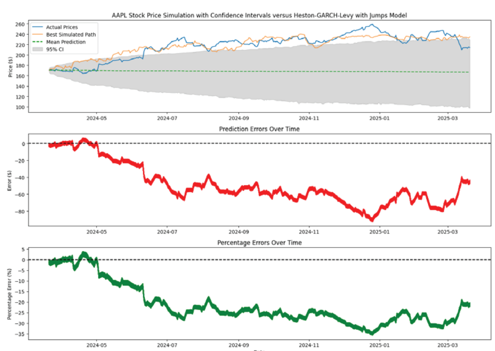

Figure 4: HGL model tested against AAPL stock prices, with percentage and numerical error charts. The maximum percentage error of the HGL occurs at 35%, and several solutions of intersections between simulated and actual prices are visually displayed above. A 99% confidence interval was used to capture the true expected simulation of all probable outcomes.

Figure 4 is an error analysis simulation of our own model, the HGL. In this case, we have a maximum error at around 35 percent, and we once again use 99 percent confidence intervals to capture the true expected simulation of our stock price dynamics. Note, I say expected as there are many possibilities for predictions. The one we choose will be the average or expected simulation of our process in the HGL model. When compared side-by-side, the HGL model has a significant improvement in approximations compared to the Black-Scholes model. The Black-Scholes model has a maximum error of 40% deviations, while the HGL model has a 35% maximum error. The HGL model is also better at tracking price behaviors, as tendencies to move in certain directions are mimicked by the HGL. To further see this, look at the figure below of a broader simulation of the HGL against AAPL prices.

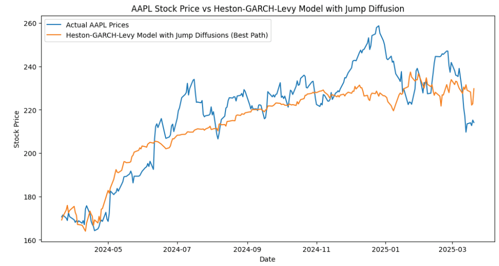

Figure 5: HGL sample simulation against AAPL prices with enhanced jump diffusions. The HGL tends to follow the growth trend of AAPL prices in early 2024, and early in the simulation, the drops and spikes and often mimicked. In real-world trading, options trading is significantly enhanced with our model's improvements on previous versions of stochastic processes.

Large Deviations in the Poisson Process

Furthermore, in discussing the resulting simulations, we must note that the large deviations matter in predicting abnormal stock price dips and spikes.[13] In section 5.2, it has been noted that the Lipschitz condition prevents the model's explosive growth (of course, with the linear growth condition allowing the existence of solutions). However, immense changes in the stock price have occurred in the past, and thus, we cannot rule out the ability to predict such instances.

The number of jumps in the stock price, defined by N(t), follows a Poisson process with:

$$P(N_t = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Using Cramér's theorem, we estimate $P(N_t > \lambda_t + \delta_t)$. We define the scaled process as $\frac{N_t}{t} = \lambda + \frac{1}{t}\sum_{i=1}^{N_t} J_i$ and we use Sanov's theorem to give $P(N_t > \lambda_t + \delta_t) \approx e^{-tI(\delta)}$, where the rate function is: $I(\delta) = \sup_{\theta}[\theta\delta - \log \mathrm{E}e^{\theta N_t}]$

For Poisson-distributed N(t), $\Lambda(\theta) = \lambda(e^{\theta} - 1)$, yielding:

$$I(\delta) = \delta \log\left(\frac{\delta}{\lambda}\right) - \delta + \lambda$$

What this states, intuitively, is that jump clustering is exponentially rare but still possible. Thus, we do not rule out a possible extreme market crash or market spike. Additionally, if J(t) follows a heavy-tailed Lévy distribution, the rate function decays more slowly, increasing the likelihood of extreme price movements.

As for the GARCH volatility process, to show that volatility can also have such extreme predictions, we can showcase the large deviations proof by rewriting the GARCH volatility dynamics as a discrete sum: $v_t = \sum_{k=0}^{\infty} \beta^k(\omega + \alpha\epsilon_{t-k}^2)$. The empirical mean is then defined as $\frac{1}{T}\sum_{t=1}^{T} v_t$. The Gärtner-Ellis theorem gives us the large deviation rate function as $I(x) = \sup_{\theta>0}[\theta x - \Lambda(\theta)]$ where $\Lambda(\theta)$ is the moment-generating function for this volatility process. For large x-values, $P(v_t > x) \approx e^{-TI(x)}$. For heavy-tailed innovations (e.g., Student-t distributions), I(x) decays more slowly, increasing the probability of extreme volatility. Thus, extreme volatility states can persist and thus make risk assessment crucially involved in trading decisions.

To best understand this from a non-mathematical standpoint, consider that while our model has a low probability of enacting high motion jump dynamics, there still "is" a probability that is on the tail end of the Poisson. At this level, the distribution values that we obtained are incredibly high, and will thus affect the price prediction significantly away from the drift function with the risk-free interest rate.

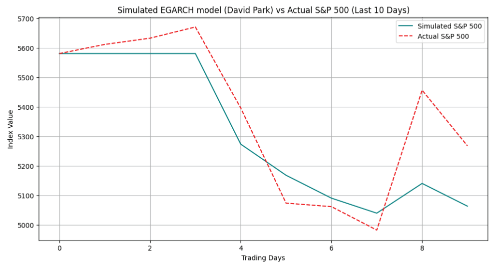

Figure 6: HGL tested against the S&P 500 for an extreme market crash under Trump tariffs. HGL simulation demonstrates mimicking patterns with the AAPL stock prices.

Figure 6 showcases an example of how our model applies to real-life scenarios. During the recent Trump administration's tariffs, the stock market experienced a period of severe decline. The HGL predicted the exact moment at which this fall begins, and thus, a put option would safely put the investor in a winning scenario despite extreme market crashes. Notice, the predictions within this time frame also match the characteristic behaviors. When the prices are falling, so does the HGL prediction, and when they shift up, so does the HGL. These characteristics are particularly powerful in options trading, where there are fundamental strategies for buying puts or calls that are easy to apply with this model's behaviors.

And so, the model is successful in incorporating extreme market events as part of its predictive capabilities and is also stable. The model's simulations are both characteristically and statistically more accurate in price predictions than previous versions and thus can be utilized effectively in options and algorithmic trading.

Conclusion

The analysis presented in this paper confirms that the stochastic differential equations (SDEs) of the Heston-GARCH-Lévy model satisfy the local Lipschitz continuity and linear growth conditions for the drift, diffusion, and jump terms. As a result, the existence and uniqueness of a strong solution to the model are established based on standard stochastic calculus theorems.[14] Beyond numerical accuracy, this study provides a theoretical foundation for the use of specific structures in financial modeling. In particular, we justified the Cox-Ingersoll-Ross (CIR) process for variance modeling, which ensures non-negative volatility paths, making it more suitable than standard Brownian motion for financial applications, as seen in ref [8]. Furthermore, we established the necessity of working within the L^2 normed space, providing counterexamples that illustrate why alternative normed spaces fail to maintain the desired mathematical properties for our SDE framework. These justifications strengthen the mathematical rigor behind the model's construction and ensure its reliability for practical applications. The Heston-GARCH-Lévy model offers a more robust and flexible framework for asset price modeling by addressing key limitations of traditional models like Heston, SABR, Rough Volatility, and Variance Gamma. Unlike these models, our approach effectively captures volatility clustering, stochastic mean reversion, and discontinuous price jumps, providing a more comprehensive reflection of financial market behaviors. However, despite its accuracy, the computational cost remains a challenge. Future work will focus on calibrating the model using historical market data and implementing it in real time for derivative pricing applications. Additionally, since Monte Carlo simulations remain computationally expensive, efforts will be made to improve simulation efficiency through optimized numerical methods, reduced variance techniques, and potentially leveraging high-performance computing for faster simulations without compromising accuracy. By refining both the theoretical and computational aspects of the model, we will aim to enhance its applicability in quantitative finance and risk management. Additionally, wavelet simulations will be attempted to cover extreme noise in different markets. Future work will also incorporate the extension of our model toward other markets, such as cryptocurrencies, with more sets of SDEs. Lastly, new rigorous proofs of stability will be studied for hybrid stochastic systems for markets outside of tech-based stocks.

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Author

David Park is a high school senior from Campbell Hall, a private school in Studio City, CA. He is a current student of the class of 2026, and he is entering the class of 2030 for his undergraduate career. He is enthusiastic about researching the depths of stochastic processes and how randomness and stochastic differential equations can model several different aspects of financial derivatives and securities. Additionally, he is passionate about the fields of Measure Theory and probability, which make up the foundation of stochastic research. His research has been acknowledged by The David J. Park Foundation for Innovation & Technology and has been granted funding to pursue the hybrid model even further in the world of computer science. He aspires to pursue the fields of financial engineering and applied mathematics moving forward, and is an eager researcher with multiple recognitions in the field of stochastic processes and their applications.