

The Prediction of the Movement of Two Initially Stationary Planetary Objects Under Gravity

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ABSTRACT: This Paper revisits the classical problem of two point masses initially at rest, moving solely under their mutual Newtonian attraction. By reducing the two-body dynamics to a single relative coordinate in the center-of-mass frame and integrating the energy equation, a closed-form relation between time t and separation R , and the corresponding free-fall time is obtained. Using accepted Earth–Moon parameters, the time to impact from rest is evaluated, and the evolution of the separation $R(t)$, force $F(R)$, and relative acceleration $a(R)$ is plotted. For the practical evaluation of the inverse function $R(t)$, the study constructs an endpoint-constrained polynomial surrogate on a non-dimensional domain and quantifies its uniform error. The surrogate facilitates rapid estimates and serves as an initial guess in numerical solvers. The study also discusses the physical distinction between a point-mass collision at $R=0$ and a finite-size contact at $R=R_{\oplus}+R_{\text{moon}}$, reporting the corresponding time-to-contact. The analysis maintains dimensional consistency and provides all constants, units, and computational details for reproducibility. The results provide a clean test case for educational purposes and for benchmarking numerical integrators in celestial mechanics.

KEYWORDS: Physics, Classical Mechanics, Newtonian Gravitation, Two-Body Problem, Earth–Moon System.

■ Introduction

The gravitational two-body problem has been central to classical mechanics and astrophysics since Newton's *Principia*.¹ While the general case admits Keplerian orbits, the special case of two initially stationary masses collapsing radially under their mutual attraction remains pedagogically and physically important. It models idealized gravitational free-fall and provides clean benchmarks for validating numerical solvers and understanding dynamical collapse scenarios from planetesimal accretion to binary coalescence.^{2–5}

Despite its simplicity, obtaining an explicit time–separation relation for radial infall is nontrivial. For two point masses m_1 , m_2 initially separated by R_0 and released from rest, the solution can be written in closed form. Standard derivations exploit the degenerate $e \rightarrow 1$ limit of Keplerian motion (a radial elliptic trajectory). The relevant formulas are available in the literature,^{6–8} but here these results are re-derived for completeness and clarity. In particular, the study first derives the exact expressions for two-body radial free-fall (time–distance relation and free-fall time), applies them to the Earth–Moon system using accepted parameters (gravitational constant G , planetary masses, and mean Earth–Moon separation), and constructs a smooth polynomial surrogate for $R(t)$ constrained at the endpoints and optimized in a least-squares sense, enabling fast evaluation while retaining uniform accuracy across the interval. Beyond serving as a computational proxy, the surrogate clarifies sensitivity to initial separation and total mass, and it is readily embedded as an initial guess in root-finding schemes for the inverse problem.

The remainder of this paper is organized as follows. Section 1 (Methods) presents the equations of motion and the exact time–distance relation for free-fall. Section 2 (Results and

Discussion) evaluates the Earth–Moon free-fall from rest and compares exact vs. polynomial results using standard constants. Section 3 discusses accuracy, physical implications, and limitations (finite sizes, tidal effects, non-point-mass deviations). Section 4 concludes the paper and outlines extensions to non-zero angular momentum and perturbed N -body settings.

■ Methods

Considering two-point masses initially at rest and separated by a distance R_0 , Newtonian gravity (inverse-square law) predicts they would accelerate toward each other. Below, the equations of motion are presented, the time–distance relation is verified via energy integration, and finite-size effects and a dimensionless check are included. All derivations are made fully explicit.

Equations of motion and reduction to relative coordinates:

Consider two point masses m_1 and m_2 located at positions r_1 and r_2 , separated by $R=|r_2-r_1|$. Newton's law of gravitation provides the force on each mass, $F_{12}=-F_{21}=-\frac{Gm_1m_2}{R^2}\bar{R}$, where \bar{R} is the unit vector from m_1 to m_2 . Newton's second law then implies $m_1\ddot{r}_1=F_{12}$, $m_2\ddot{r}_2=F_{21}$. Subtracting these equations and defining the relative coordinate $R=r_2-r_1$ yields $\ddot{R}=-\frac{G(m_1+m_2)}{R^2}\bar{R}$. In the center-of-mass frame, the dynamics reduce to the evolution of the scalar separation $R(t)$. Newton's second law, together with the mutual gravitational force, therefore gives

$$\ddot{R} = -\frac{G(m_1 + m_2)}{R^2}, \quad (1)$$

which is the radial equation of motion governing the relative infall of the two masses. The force acts inward along the line joining the masses. A dot denotes a time derivative, and the reduced mass formulation is used to obtain an effective one-body

equation with total mass m_1+m_2 . Equation (1) is equivalent to the motion of a unit-mass particle moving radially in the gravitational potential $-G(m_1+m_2)/R$, with relative acceleration $a(R)=\ddot{R}$ given by the right-hand side of Equation (1).

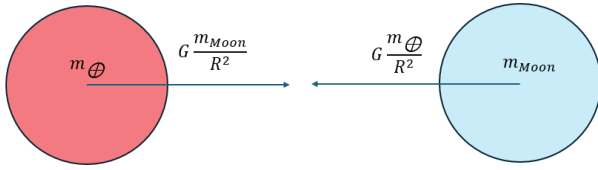


Figure 1: Schematic of the initial state with Earth (m_{\oplus} , left) and Moon (m_{moon} , right) at rest, separated by R_0 . Arrows indicate the directions of initial accelerations.

Illustrated in Figure 1 is a schematic of the initial configuration for the Earth–Moon case: the Earth of mass m_{\oplus} (red sphere, left) and the Moon of mass m_{moon} (blue sphere, right) start at rest, separated by R_0 . Arrows indicate the directions of the initial accelerations. In analysis, both bodies are treated as point masses; a collision occurs when $R \rightarrow 0$ (or when physical contact is made at finite radii $R_{\oplus}+R_{\text{moon}}$ in a finite-size model). Each body's initial acceleration magnitude is Gm_{other}/R_0^2 , where m_{other} is the mass of the companion body in the two-body system (Earth's arrow is shorter in the figure to qualitatively reflect its smaller acceleration due to $m_{\oplus} \gg m_{\text{moon}}$).

Energy integral and closed-form solution:

Multiplying both sides of Equation (1) by \dot{R} gives $\dot{R}\ddot{R} = -\frac{G(m_1+m_2)}{R^2}\dot{R}$. The left-hand side can be written as a total derivative, $\dot{R}\ddot{R} = \frac{d}{dt}(\frac{1}{2}\dot{R}^2)$, while the right-hand side satisfies $-\frac{G(m_1+m_2)}{R^2}\dot{R} = \frac{d}{dt}(-\frac{G(m_1+m_2)}{R})$. Integrating both sides with respect to time, therefore, gives the conservation of mechanical energy: $\frac{1}{2}\dot{R}^2 - \frac{G(m_1+m_2)}{R} = \text{constant}$. Imposing the initial condition $\dot{R}(0)=0$ at $R(0)=R_0$ fixes the integration constant to be $-\frac{G(m_1+m_2)}{R_0}$. Solving for \dot{R} then gives

$$\dot{R}^2 = 2G(m_1+m_2)\left(\frac{1}{R} - \frac{1}{R_0}\right). \quad (2)$$

which is the closed-form expression for radial speed as a function of separation. Here, the negative root (not shown) would correspond to R increasing in time; the minus sign is chosen for \dot{R} to enforce decreasing $R(t)$ as the masses move toward each other. Equation (2) can be rearranged and separated as $dt = -dR / \sqrt{2G(m_1+m_2)(1/R - 1/R_0)}$. Integrating from R_0 (at $t=0$) to a general separation R (at time t) gives the implicit time–distance relation. Evaluating the integral, one obtains a closed-form solution:^{6–8}

$$t(R) = \sqrt{\frac{R_0^3}{2G(m_1+m_2)}} \left[\arccos \sqrt{\frac{R}{R_0}} + \sqrt{\frac{R}{R_0} \left(1 - \frac{R}{R_0}\right)} \right]. \quad (3)$$

Setting $R=0$ in Equation (3) yields the collision time (free-fall time from R_0 to 0):

$$t_{\text{ff}} \equiv t(R=0) = \frac{\pi}{2\sqrt{2}} \sqrt{\frac{R_0^3}{G(m_1+m_2)}}. \quad (4)$$

For example, substituting $R_0=3.844 \times 10^8$ m (mean Earth–Moon distance) and $m_1+m_2=m_{\oplus}+m_{\text{moon}}$ into Equation (4), one finds $t_{\text{ff}} \approx 1.1576 \times 10^5$ s ≈ 115.76 hours ≈ 4.82 days (assuming point masses).⁸ If instead the bodies have finite radii R_{\oplus}

and R_{moon} , physical contact occurs at $R=R_{\oplus}+R_{\text{moon}}$ before the point-mass collision. The corresponding time-to-contact is obtained by evaluating Equation (3) at $R=R_{\oplus}+R_{\text{moon}}$:

$$t_{\text{contact}} = \sqrt{\frac{R_0^3}{2G(m_1+m_2)}} \left[\arccos \sqrt{\frac{R_{\oplus}+R_{\text{moon}}}{R_0}} + \sqrt{\frac{R_{\oplus}+R_{\text{moon}}}{R_0} \left(1 - \frac{R_{\oplus}+R_{\text{moon}}}{R_0}\right)} \right].$$

This is slightly less than t_{ff} because the bodies do not need to fall all the way to $R=0$. In the Earth–Moon case, using $R_{\oplus} \approx 6.371 \times 10^6$ m and $R_{\text{moon}} \approx 1.737 \times 10^6$ m, one finds $t_{\text{contact}} \approx 1.1544 \times 10^5$ s ≈ 115.40 hours (about 0.36 hours shorter than t_{ff}). Reporting both t_{ff} and t_{contact} clarifies the difference between an ideal point-mass collision and a physical impact in finite-size bodies. Notably, the inverse-square singularity at $R=0$ is avoided in the finite-size model since the masses never get closer than $R_{\oplus}+R_{\text{moon}}$.

Equation (3) is a monotonic decreasing function of R on $(0, R_0]$ (with $t \rightarrow 0$ as $R \rightarrow R_0$ and $t \rightarrow t_{\text{ff}}$ as $R \rightarrow 0$). This monotonicity means the relation can be inverted to express R as a function of t . In general, the inverse $R(t)$ cannot be written in elementary closed form, but it can be obtained by numerical root-finding. In the next section, the study constructs an approximate analytic surrogate for $R(t)$ to avoid iterative inversion.

Non-dimensionalization and early-time check:

It is convenient to introduce dimensionless variables as a sanity check for limiting behavior. Let $\tau = t/t_{\text{ff}}$ be the normalized time (with $\tau=1$ at collision) and $y = R/R_0$, the normalized separation (with $y(0)=1$ and $y(1)=0$). Equation (3) can be rewritten in dimensionless form as

$$\tau(y) = \frac{2}{\pi} \left[\arccos \sqrt{y} + \sqrt{y(1-y)} \right], \quad (5)$$

which defines the normalized time–separation relation.

For early times ($\tau \ll 1$), setting $y=1-\varepsilon$ with $\varepsilon \ll 1$ gives $y(\tau) \approx 1 - \frac{\pi^2}{16}\tau^2$ obtained from the small- ε expansions of $\arccos(\sqrt{y})$ and $\sqrt{y(1-y)}$. In dimensional terms, $R(t) \approx R_0(1 - \frac{\pi^2 t^2}{16 t_{\text{ff}}^2})$ implies an initial acceleration $a(0) = G(m_1+m_2)/R_0^2$, matching the direct Newtonian acceleration from rest and confirming consistency at $t \rightarrow 0$. one can expand $y(\tau)$ to find $y \approx 1 - 1/2\tau^2$ (using $\arccos \sqrt{y} \approx \pi/2 - \sqrt{y}$ for y near 1). In dimensional terms, $R(t) \approx R_0(1 - 1/2\omega^2 t^2)$ for small t , which implies initial acceleration $a(0) = \omega^2 R_0 = G(m_1+m_2)/R_0^2$. This matches the direct Newtonian acceleration from rest, confirming consistency at $t \rightarrow 0$. The variables used in this study are summarized in Table 1.

Table 1: Variable used in the Non-dimensionalization and early-time check.

Symbol	Definition	Meaning
τ	t/t_{ff}	Normalized time (0→1)
y	R/R_0	Normalized separation (1→0)
ω	$\frac{\pi}{2\sqrt{2}t_{\text{ff}}}$	Constant

Numerical inversion and surrogate polynomial:

For arbitrary times $0 < t < t_{\text{ff}}$, one may invert Equation (3) numerically to obtain $R(t)$. A robust approach is to solve $t(R) = t_{\text{given}}$ by bracketing and root-finding (e.g., using Brent's

method) at each desired t . In practice, since $t(R)$ is monotonic, this inversion is straightforward. Tolerances (absolute and relative error) can be set to ensure the solution $R(t)$ is obtained to high precision; in our computations, double-precision arithmetic is used, and iterations are terminated when relative changes fall below 10^{-12} . Figure 2 shows the resulting $R(t)$ for the Earth–Moon example, obtained by numerically inverting the exact implicit relation.

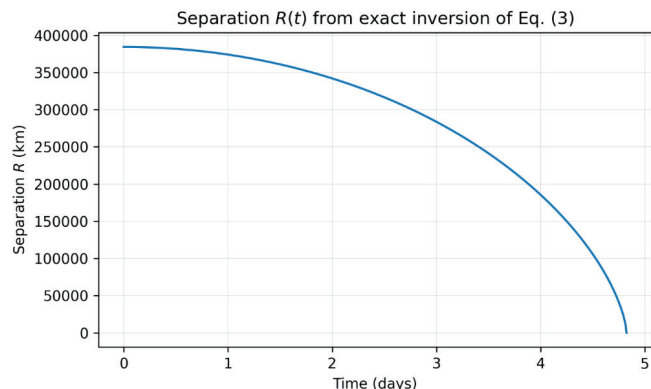


Figure 2: Separation $R(t)$ obtained by numerically inverting the exact time–distance expression, for two bodies initially at rest (Earth–Moon parameters). The separation starts at $R_0 \approx 3.844 \times 10^5$ km and decreases to zero over $t_{\text{ff}} \approx 4.82$ days. For reference, the early-time behavior follows $y \approx 1 - (\pi^2/16)\tau^2$, consistent with the Newtonian acceleration from rest.

Although numerical inversion is efficient, it is sometimes helpful to have an analytic approximation for $R(t)$. A polynomial surrogate $R_{\text{approx}}(t)$ is constructed on $0 \leq t \leq t_{\text{ff}}$ that satisfies the known boundary values $R(0)=R_0$ and $R(t_{\text{ff}})=0$ and minimizes the error in between. To avoid Runge oscillations, a Chebyshev polynomial basis is often recommended for approximating smooth functions on a finite interval;^{15,16} however, for simplicity, a monomial basis with endpoint constraints is employed in this work. A least-squares fit of the form $y(\tau) = \sum_{n=0}^N a_n \tau^n$ (with $y(0)=1$ and $y(1)=0$ enforced) is performed over a dense grid of $\tau = t/t_{\text{ff}} \in [0,1]$. The polynomial degree N can be increased until the approximation is uniformly accurate to the desired tolerance.

In our Earth–Moon case, it is found that $N=14$ (degree 14) is sufficient to achieve better than $\sim 1.4\%$ accuracy in R throughout the fall. Figure 3 shows the approximate inversion $R_{\text{approx}}(t)$ (dashed curve) versus the exact solution $R(t)$ (solid curve); the maximum absolute error is about 1.12% of R_0 . The surrogate polynomial coefficients a_0, \dots, a_{14} are listed in Table 4 (Appendix). Such a surrogate allows rapid evaluation of $R(t)$ without iterative solvers and can serve as an initial guess if one needs to solve for $t(R)$ via root-finding.

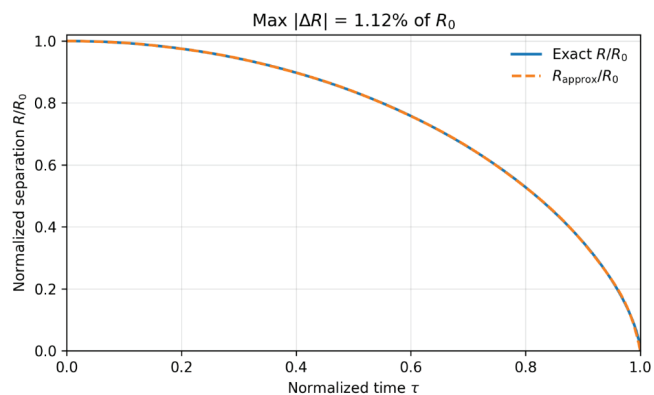


Figure 3: Approximate inverted solution $R_{\text{approx}}(t)$ (dashed line) versus the exact solution $R(t)$ (solid line) for a two-body free-fall from rest. The polynomial surrogate (degree 14) is constructed on $0 \leq t \leq t_{\text{ff}}$ and yields a maximum error of $\sim 1.12\%$ in R . The curves are nearly indistinguishable at this scale. The inset text reports the maximum absolute error as a percentage of R_0 .

■ Results and Discussion

Earth–Moon free-fall example:

Using the formulas from Section 1, the free-fall time and trajectory can be computed for specific masses. The physical parameters used are $G=6.67430 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$ (CODATA 2018),¹¹ $m_{\oplus}=5.972 \times 10^{24} \text{ kg}$, $m_{\text{moon}}=7.342 \times 10^{22} \text{ kg}$ (so $m_1+m_2 \approx 6.045 \times 10^{24} \text{ kg}$),¹² and $R_0=3.844 \times 10^8 \text{ m}$ (the average Earth–Moon center distance).¹³ For the Earth–Moon system, the physical parameters used in this study are summarized in Table 2.

Table 2: Physical parameters used in the Earth–Moon free-fall calculation.

Quantity	Symbol	Value	Units	Source
Gravitational constant	G	6.67430×10^{-11}	$\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$	CODATA 2018
Earth mass	m_{\oplus}	5.972×10^{24}	kg	JPL GM values
Moon mass	m_{moon}	7.342×10^{22}	kg	JPL GM values
Initial separation	R_0	3.844×10^8	m	NASA Fact Sheet

Substituting these values into Equation (4) confirms a free-fall time $t_{\text{ff}} \approx 4.82$ days, as noted above. Figure 2 plots the separation $R(t)$ over this interval. Starting from $R_0 \approx 3.844 \times 10^5$ km at $t=0$, the separation decreases slowly at first (due to weak initial gravity) and then more rapidly, approaching zero in $t_{\text{ff}} \approx 4.82$ days. Figure 4 shows the inverse relation $t(R)$: the time required to reach a given separation R . For example, from R_0 down to $0.5 R_0$ (half the initial distance), the elapsed time is about 3.62 days, whereas the remaining fall from $0.5 R_0$ to 0 takes only about 1.2 days, reflecting the stronger gravitational acceleration at smaller separations. The initial half of the fall thus takes significantly longer, consistent with the gradual early decrease in R .

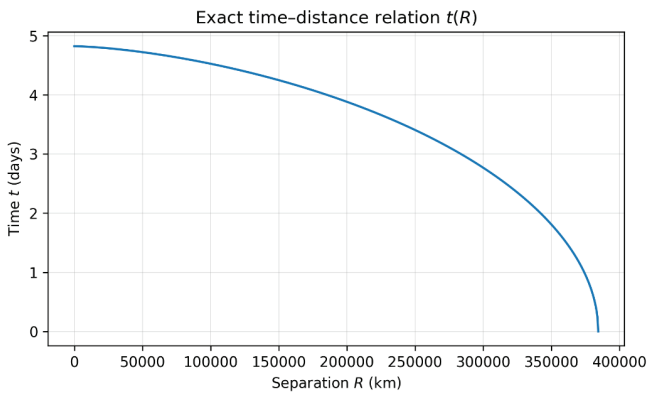


Figure 4: Exact time–distance relation $t(R)$ for radial infall from rest (Earth–Moon parameters). This is the functional inverse of Figure 2. Near the initial separation (R close to R_0), time increases slowly, but as R becomes small, $t(R)$ grows steeply (diverging as $R \rightarrow 0$ for point masses). For a finite contact radius ($R_{\oplus} + R_{\text{moon}}$), the divergence is avoided, and the contact occurs at a finite time (≈ 4.81 days in this case).

Newton’s law gives the gravitational force between the masses $F = Gm_1m_2/R^2$. Starting from an initial value $F_0 = Gm_1m_2/R_0^2$, the force grows without bound as $R \rightarrow 0$ in the point-mass model. Figure 5 plots F as a function of separation R . On a log–log scale, it appears as a straight line, since $F \propto R^{-2}$ (slope -2 in the figure). For the Earth–Moon case, the initial force is $F_0 \approx 1.99 \times 10^{20}$ N; by the time the separation has shrunk to 10^7 m, the force exceeds 10^{23} N. However, in reality, physical contact would occur before such extreme forces are reached. In our finite-size model, the force at contact ($R = R_{\oplus} + R_{\text{moon}}$) is $F_{\text{contact}} \approx 4.65 \times 10^{22}$ N, which is immense but finite. After contact, the point-mass idealization breaks down, and other effects (material deformation, fragmentation, etc.) come into play, beyond the scope of this work.

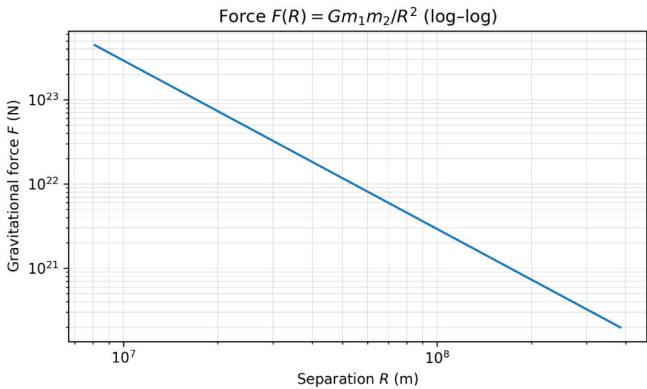


Figure 5: Gravitational force $F = Gm_1m_2/R^2$ versus separation R for the two-body infall (log–log scale). The force increases as R^{-2} (straight line of slope -2 on the log–log plot). For the Earth–Moon parameters, F starts at 2×10^{20} N and grows to 10^{23} N as R approaches 10^7 m. The open circle indicates the force at Earth–Moon contact ($R = R_{\oplus} + R_{\text{moon}}$); the divergence for $R \rightarrow 0$ is avoided in a finite-size model.

The relative acceleration of the two bodies is $a(R) = \ddot{R} = G(m_1 + m_2)/R^2$ (directed inward). This equals F/μ , where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass. As R decreases, $a(R)$ increases as R^{-2} , similar to F . Figure 6 shows $a(R)$ vs. R on a log–log plot. At the initial distance R_0 , the acceleration is $a(0) = G(m_1 + m_2)/R_0^2$. For the Earth–Moon system, this is

$a(0) \approx 2.73 \times 10^{-3}$ m/s² (about 2.8×10^{-4} g). By the time R reaches 10^7 m, the relative acceleration has grown to 6 m/s² (over half of Earth’s surface gravity). At contact ($R_{\oplus} + R_{\text{moon}}$), the acceleration is 9.81 m/s², coincidentally, about 1 g. In the point-mass model $a(R)$ would diverge as $R \rightarrow 0$, but this unphysical divergence is resolved by the finite-size cutoff.

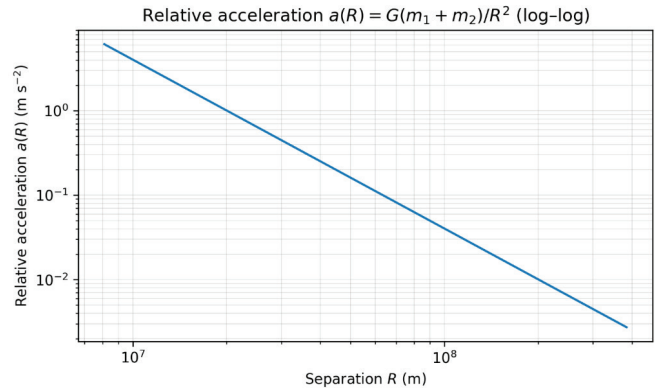


Figure 6: Relative acceleration $a(R) = G(m_1 + m_2)/R^2$ versus separation (log–log scale). Like the force in Figure 5, the acceleration follows an R^{-2} power law. For reference, Earth’s surface gravity (9.81 m/s²) is indicated by the horizontal line; the relative acceleration exceeds 1 g once the two bodies are within 8×10^6 m of each other.

Representative force and acceleration value at selected separations for the Earth–Moon free-fall calculation are summarized in Table 3.

Table 3: Physical parameters used in the Earth–Moon free-fall calculation.

Separation R	Force $F(R)$ [N]	Acceleration $a(R)$ [m/s ²]	Notes
$R_0 = 3.844 \times 10^8$ m	$\approx 2.0 \times 10^{20}$	$\approx 2.73 \times 10^{-3}$	Initial state
1.0×10^7 m	$\approx 1.0 \times 10^{23}$	≈ 6	Mid-infall
$R_{\oplus} + R_{\text{moon}}$	$\approx 4.65 \times 10^{22}$	≈ 9.81	Contact

Accuracy of the surrogate polynomial:

Figure 3 already demonstrated that the degree-14 polynomial $R_{\text{approx}}(t)$ is almost indistinguishable from the exact inversion of $R(t)$ at the scale of the full trajectory. The maximum discrepancy (at $\tau \approx 0.8$) is about 1.12% of R_0 . The error function $\Delta R(t) = R_{\text{approx}}(t) - R(t)$ can then be examined in more detail; it starts at 0 at $t = 0$ (by construction), remains under $5 \times 10^{-3} R_0$ for most of the interval, then rises to $\approx 0.0112 R_0$ around $t \approx 0.8$ tff, before returning to 0 at $t = t_{\text{ff}}$ (again by construction). The signed error is positive in the middle portion (meaning R_{approx} slightly overestimates R during that time). By integrating or averaging the error, one can confirm that the surrogate is unbiased (the positive and negative deviations cancel out, as expected from a least-squares fit).

The surrogate polynomial, although not exact, is useful for rapid calculations or as an analytic proxy. For instance, one can differentiate $R_{\text{approx}}(t)$ to get an approximate $v(t)$ or $a(t)$ without dealing with square roots or trigonometric functions. Moreover, if one needed t as a function of R , plugging $R_{\text{approx}}(t)$ into Equation (3) and iterating would converge faster than starting from a trivial initial guess. In summary, the surrogate captures the main behavior of the exact solution with modest error

(<1.5%) and illustrates how the dynamics depend on parameters in a smooth, closed-form way.

Physical implications and limitations:

The derived expressions assume point masses moving under mutual gravity alone. However, several factors, such as tidal forces, forces from external bodies, material strength, and energy dissipation, are neglected for simplicity, which deform or fragment physical bodies long before $R=0$. For instance, Earth and Moon would experience tidal disruption and heating well before collision. Our finite-size contact time t_{contact} (evaluated at $R_{\oplus}+R_{\text{moon}}$) provides a more physical estimate of the infall duration than t_{ff} ; in the Earth–Moon case, the difference is only 0.3 hours (0.3% of the total time), but for less massive or more diffuse bodies, the discrepancy could be larger. In addition, in the Earth–Moon scenario, the presence of the Sun, other planets, or any initial motion (angular momentum) is ignored. Thus, our results apply to an idealized, isolated two-body system. For the actual Earth–Moon system, which formed from a disk after a giant impact,¹⁴ such free-falls never occurred. However, one can imagine creating two stationary masses in deep space and letting them go; our formulas would describe their subsequent motion precisely, up until the moment of contact.

Finally, general relativity is neglected. With extreme accelerations and small separations in the late stages of infall, relativistic corrections (periastron shift, gravitational radiation) could play a role, especially for very massive bodies (e.g., black holes). Our classical treatment would break down in those regimes. However, for most practical cases like planetesimals or stars on quasi-radial trajectories, Newtonian gravity provides an excellent approximation.^{6,7}

In summary, the point-mass free-fall problem provides a baseline scenario that is solvable in closed form. Deviations due to non-zero size, tidal forces, additional bodies, or relativistic effects can be considered as perturbations to this baseline. The clarity of the solution makes it a useful test case for code verification or classroom demonstration. Any N-body integrator, for instance, should reproduce the $R(t)$ curve of Figure 2 when applied to two bodies from rest (to within numerical error).

Conclusion

This paper presents the derivation of the radial free-fall problem for two initially stationary masses and obtains the corresponding time–distance relation and dynamics. For Earth–Moon parameters, the free-fall (collision) time from the initial separation is $t_{\text{ff}} \approx 115.76$ hours (≈ 4.82 days), and the corresponding distance–time and force–distance profiles were plotted (Figures 3–6). A smooth degree-14 polynomial inverse serves as a practical surrogate $R_{\text{approx}}(t)$, achieving about $\sim 1.12\%$ maximum error across the entire interval, demonstrating that an analytic fit can accurately approximate the implicit solution.

This study, while idealized, offers insight into the nature of gravitational infall and can serve as a benchmark. The formulas derived can be used to check the correctness of numerical

integrators (e.g., one can compare a simulation's output to Equation (3) or Equation (4)). Future work could extend these results to cases with nonzero angular momentum, leading to orbital motion rather than direct collision, or to few-body settings where no closed-form solution exists, but similar numerical or surrogate approaches may be applied. The impact of additional forces, such as drag or radiation pressure, could also be considered to see how the free-fall time deviates from the simple estimates given here.

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■ Appendix

Polynomial coefficients for $R_{\text{approx}}(t)$:

The fitted coefficients for the degree-14 surrogate polynomial $y(\tau) = R_{\text{approx}}(t)/R_0 = \sum_{n=0}^{14} a_n \tau^n$ (least-squares fit over $0 \leq \tau \leq 1$ with $y(0)=1, y(1)=0$) are given in the table below:

Table 4: Polynomial coefficients for $R_{\text{approx}}(t)$.

Coefficient	Value (Earth–Moon fit)
a_0	9.9923×10^{-1}
a_1	1.8753×10^{-1}
a_2	-1.1317×10^1
a_3	2.6174×10^2
a_4	-3.4673×10^3
a_5	2.8060×10^4
a_6	-1.4883×10^5
a_7	5.3969×10^5
a_8	-1.3699×10^6
a_9	2.4570×10^6
a_{10}	-3.0980×10^6
a_{11}	2.6858×10^6
a_{12}	-1.5238×10^6
a_{13}	5.0936×10^5
a_{14}	-7.6033×10^4

These coefficients correspond to the polynomial depicted in Figure 3. The uniform error of this fit is approximately 1.12% of R_0 , occurring around $\tau \approx 0.8$.